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EDITORS' NOTE: The following fourteen papers were read at a Symposium on Quantum Mechanics, held under the auspices of the American Physical Society in New York City, December 31, 1928. It is much regretted that the editors have not been able to procure two other papers which were read on that occasion.

PHYSICAL MEANING OF WAVE MECHANICS.

BY

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AN understanding of the physical meaning of wave mechanics is essential if one is going to do useful work in the subject; even purely mathematical research in quantum theory is of small value unless it is carried out with the proper understanding of the physics behind it. For that reason, it seemed well to start this discussion with a brief treatment of physical interpretation rather than mathematical details. Of course, only a small portion of so large a subject can be taken up; and I have chosen to speak about the statistical side of wave mechanics, both because this is one of its most fundamental sides, and because it is a topic that will fit in well with the other papers.

Wave mechanics is an extension, not of ordinary Newtonian mechanics, but of statistical mechanics; and this simple observation is enough to explain many of its otherwise puz-

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zling features. Since ordinary statistical mechanics is a subject about which there is much misunderstanding, it will be well to start with a little description of it. Being statistical, it must deal with a great many similar objects or events; but it is necessary at the outset to understand that this by no means limits us to problems where, as in the theory of gases, there are a great many identical physical systems. One can perfectly well treat statistically a problem with even one single degree of freedom, as for example an oscillator. The statistics comes in in the fact that one works, not with single observations, but with ensembles of observations. By an ensemble, one means a set of repetitions of the same experiment—just such a set of repetitions as one actually makes in working a real physical experiment. One can set up a simple mathematical picture of an ensemble of observations. Suppose we have a space with a number of dimensions to coincide with the number of measurements we make; for example, if our system is a mechanical one with n degrees of freedom, there will be $2n$ dimensions, n for the coördinates of the system, the other n for its velocities or momenta. Each time we make our measurements, we can represent the result by a single point in this space. Then if we repeat the experiment many times, we get as many different points as there are repetitions, and this swarm of points is the mathematical picture of the ensemble. In most cases, we wish to suppose the number of trials to increase without limit, so that the swarm approaches a continuous distribution, and we can define a function representing the density of points in our space. This distribution function then will also serve as a mathematical representation of the ensemble.

The particular sort of ensemble to be used in any case depends on the physical conditions. We must judge, as closely as possible, what sort of distribution of the quantities we are measuring will really occur in our experiment, and set up an ensemble accordingly. Here, as in every case, the purpose of our mathematics is to run, so to speak, parallel to the physics; one can imagine them written in parallel columns, with a definite correspondence between, such that wherever we can make a physical observation, it can be matched with some feature of the mathematics. Then it is plain why we

need ensembles: to match the physical process of repetition. And the task of setting up the proper one must definitely be one of finding a parallel to the physics.

For example, suppose that our system were a linear oscillator; that we first took it, without energy; at an arbitrary time gave it a definite energy, the same in each repetition of the experiment; and then at a later time measured its coördinate and momentum, plotting them in a two-dimensional space. Then the ensemble of repetitions of this measurement will be easy to describe; the points will all correspond to the same energy, and will all lie on a curve in the two-dimensional space whose equation is energy = constant (an ellipse, since our system is an oscillator). But, on account of the arbitrary time when the energy was communicated, the phase will be different each time the measurement is made, and the points of the swarm will be distributed in a uniform fashion about the ellipse, according to a law which could be readily found. This ensemble is well known in statistical mechanics; it is the microcanonical ensemble. It is useful in thermodynamics, when we wish to work with systems of constant energy.

As another example, we might take our same oscillator, but expose it to different external circumstances. We now couple it, with small forces of interaction, to a system with a great many degrees of freedom, and a definite energy. At a definite time, we measure its coördinate and momentum. In this case, the oscillator can acquire any energy by interaction, and sometimes it will have one value, sometimes another, but the relative chances of the different values will be distributed according to a law which can be calculated. As before, the phases will be arbitrary. The swarm of particles in the ensemble will then cover the whole of this two-dimensional space, with a density that can be definitely found. This is the canonical ensemble. It is particularly useful in thermodynamics, because it represents what one means by a system at constant temperature; that is, a system capable of interchanging energy with a much larger system—temperature bath—of fixed properties.

The two ensembles we have mentioned are the most useful in the application of statistical mechanics to thermo-

dynamic problems; but statistical mechanics is a much wider subject in its possibilities, and an indefinite number of other ensembles may be useful. Suppose we still take our oscillator; but let us now give it approximately a definite energy, with approximately a definite phase, the precise values varying from one repetition of the experiment to another. Then, if we measure the coördinate and momentum at a definite time, all the points in our swarm will be clustered together in a small region of the two-dimensional space. This is the sort of ensemble that could be used to describe a measurement, almost definite, that was subject to small errors. The amount of spreading of the swarm would be directly connected with ideas of probable error.

Wave mechanics, being a form of statistical theory, operates with ensembles. They are represented mathematically by giving the distribution function representing the swarm of points. Thus, the familiar $\psi\psi^*$ gives the density of points in a space in which the various coördinates of the system are plotted. This is, it is true, different from the space usually used in ordinary statistical mechanics, where coördinates and momenta both are plotted. But the difference is not essential, from the present point of view, and it is easy to show the relation of the two methods. There are some other differences; we can not, for example, restrict all the points of the ensemble to too small a region, or we meet the difficulty of the principle of indeterminateness. But the essential principles are the same. Just as in ordinary statistical mechanics, we must here choose the ensemble, determined by ψ, by considering the sort of statistical distributions actually present in the repetitions of the experiment being performed.

An ensemble, in statistical mechanics, takes the place of a single set of measurements in ordinary mechanics; and as such it is but the beginning of the problem. Our real task is, for example, to trace what happens to the system as time goes on. Given an initial ensemble, we wish the final ensemble. This is where mechanics enters the problem; and here a characteristic and important difference in method between classical mechanics and wave mechanics appears. In classical statistical mechanics, we consider each separate

point of the original ensemble, representing a system with definite initial conditions; find its behavior according to the principles of Newtonian mechanics; derive in this way a final point from each initial one, and so a final ensemble from the initial one. The result, of course, depends on the sort of external forces to which the system is subjected. For example, if we have the microcanonical ensemble, and expose the system only to the internal forces natural to the oscillator, then it can be easily shown that, although the individual points of the swarm move, the ensemble as a whole remains unchanged. This would not be true if an external force were present. With a canonical ensemble, we are most interested in finding what happens when the system interacts with the temperature bath. Here each point moves so as to change not only its phase, but also its energy, with time; yet the ensemble is so chosen that it remains unchanged as a whole. Both these ensembles, since they remain constant under the action of forces, are useful in the theory of steady states. The third ensemble we have mentioned, in which all the points were concentrated close together, naturally behaves differently; as these systems are exposed to forces, all the points of the ensemble travel together, so that after the lapse of an interval of time, the ensemble will consist of a concentrated set of points in some new parts of the space.

The method by which the wave mechanics investigates the change of the ensemble with time is essentially different. The problem, briefly, is this: given one ensemble, find another; given one function, find another. The wave mechanics achieves this directly, without intermediate steps. A thing which converts one function into another is an operator; hence the importance of operator theory in wave mechanics. Functions are conveniently considered in a space of an infinite number of dimensions where each point represents a function; in this space, an operator changes one point into another. All the operators we are interested in are linear operators in function space, and this explains the importance of linear transformations and matrix theory, which is bound up directly with them, in wave mechancs. As is well known, it is not the distribution functions with which we work directly, but the wave function ψ. For

example, in the particular case of the change of the ensemble with time, we find the time rate of change of ψ as a certain operator acting on ψ:

$$-\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = H\psi.$$

Thus the progress of the system is shown by the path of a point in function space with time, just as in ordinary mechanics it is shown by the path of a point in ordinary space with time. If our problem is as simple as, given the distribution function at one time, to calculate it at a later time, we need merely take the point representing the initial ψ; follow its motion by Schrödinger's equation until the later time; and the final point will give the distribution function of the ensemble at the final time.

But more often we have a more complicated problem where our final distribution function is not to be merely a function of the same variable as the initial one, only at a later time, but is to be a function of different variables. For example, we may observe the distribution of positions of particles to start with; our final measurement may be the distribution in energy. In a case like this, it proves that the process of finding the distribution with respect to a new quantity means really finding the representation of the same function ψ with respect to new coördinates in function space. If, for example, the system is a quantized one, and we wish to find the distribution in energy, we refer to new "coordinates" in which the "unit vectors" are the orthogonal functions found in Schrödinger's theory. The distribution function, as referred to energy, is a discrete one: it consists of a value $c_n c_n^*$ giving the probability that the energy is that of the nth stationary state. The set of $c_n c_n^*$'s as a function of n is as truly a distribution curve, describing an ensemble, as is $\psi\psi^*$ as a function of x, and one form depends directly on the other, as we can see from the relation

$$\psi = \Sigma(n) c_n u_n,$$

where the u_n's are the orthogonal functions mentioned above. The general theory of changing from one set of variables to another, in this way, is the transformation theory of Dirac

and Jordan, and we see that it is just part of the general scheme of wave mechanics, in finding a distribution function for the final ensemble from that of the original one.

It is plain from what has been said that the statistical side of wave mechanics is a very important side. To get full advantage of the methods of the subject, the statistical feature should be constantly kept in mind. Problems should be followed through mathematically as they are worked physically; by looking at them in this broad way, many errors and uncertainties will be avoided.

The Magnetic Work of the United States Coast and Geodetic Survey. D. H. HAZARD. (*Ter. Mag.*, Sept., 1928.) In connection with its proper work the Coast and Geodetic Survey has taken observations of the magnetic elements so that now there are available for 5,000 places in the United States the values of the declination, inclination and horizon-intensity. Besides this at intervals of five years the observations have been repeated at a part of the stations and the results have been charted for 1905, 1915 and 1925.

The Survey operates also five widely separated magnetic observatories. "On inspecting the photographic records from an observatory the first impression is that irregular change is the predominant feature, one day's record differing from the next, sometimes to a marked degree. If the records for the less disturbed days are compared, however, it will be seen that there are certain general features which repeat themselves with considerable regularity, maximum and minimum values of fairly definite amounts occurring at fairly definite times of the day. By combining month by month the hourly values of those less disturbed days, diurnal-variation tables may be prepared showing how great a departure from the daily mean may be expected at any hour of the day on days of that type." It is thus made manifest that in different latitudes diurnal variations differ both in amplitude and phase, and that, depending on the season of the year, there is a further difference that affects the amplitude of the daily variation as well as the time when the maxima and minima occur. The declination curves are much alike all having an easterly extreme a couple of hours before noon and a westerly extreme one or two hours after noon. Sitka, the most northerly station, shows the greatest summer-range and the least winter-range. This, coupled with the fact that in general the main part of the variation occurs during daylight, seems to indicate some connection between the earth's magnetism and sunshine.

The dip curves change from station to station not only in details but in general character. "At Honolulu the predominant feature is a minimum about noon, whereas at Sitka it is a maximum about noon. The amplitude of the diurnal variation based on the less-disturbed days is not large, of the order of ten minutes of arc in declination, two minutes of arc in dip and twenty-five gammas (one gamma = 0.00001 c.g.s. unit) in horizontal intensity and vertical intensity." At times of magnetic storms the departure from the mean may amount to two degrees in declination, and to 500 gammas in the intensities. G. F. S.

WAVE MECHANICS.

BY

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I TAKE it that, in its ideal form, the fundamental problem of atomic physics, in the spirit of the wave mechanics at any rate, is this: We have a number of entities, such as electrons, hydrogen atoms, water molecules, and so forth. Each of these is specified by a certain number of coördinates. For example, one may be specified by coördinates $q_1, q_2, \cdots q_{n_1}$, another by coördinates $q_{n_1+1}, \cdots q_{n_2}$, a third by $q_{n_2+1}, \cdots q_{n_3}$, and so on, and we wish to formulate the laws of nature as regards all the experiments we do and things we observe, in the following way: We wish to discover as a function of the q's, a quantity

$$\psi = \psi(q_1, q_2 \cdots t)$$

such that the probability that the coördinates of any one of the entities specified by, say, $q_{n_p}, q_{n_p+1}, \cdots q_{n_s}$ shall lie between $q_{n_p}, q_{n_p+1} \cdots q_{n_s}$, and $q_{n_p} + dq_{n_p}, q_{n_p+1} + dq_{n_p+1}, \cdots q_{n_s} + dq_{n_s}$, irrespective of the values of the other q's, is

$$(dq_{n_p}, dq_{n_p+1}, \cdots dq_{n_s}) \int\int\int \cdots \int\int \psi\bar{\psi} dq_1 dq_2 \cdots,$$

where the variables with respect to which the integral is taken include all of those whose differentials do not occur in the factor in parenthesis outside of the integral, where the integral itself is taken for all positive and negative values of the variables of integration, and where $\bar{\psi}$ is the conjugate to

[1] We may well raise the question as to what distinguishes one entity from another. How does the mathematics know that certain of the coördinates are to be associated with one entity and certain others with another entity? Perhaps the most logical view to take is one in which we postulate that the coördinates may be divided off into groups for which the statement in the text is true. These groups then serve to *define* the separate entities. Our mental picture of the distinctness of one hydrogen molecule from another is so vivid that we are apt to forget that the analysis requires any criterion for that distinctness.

ψ.[1] Moreover, we wish to discover the function ψ as a solution of a differential equation.

$$\text{Expression involving } \psi, \text{ its derivatives, the coördinates, and the time } t = 0,$$

of which the initial, or appropriate boundary conditions shall be the things whose assignment represents the expression of the particular experiment which is being carried out with the entities, or the particular postulated condition of the entities whose consequences we are interested in tracing.

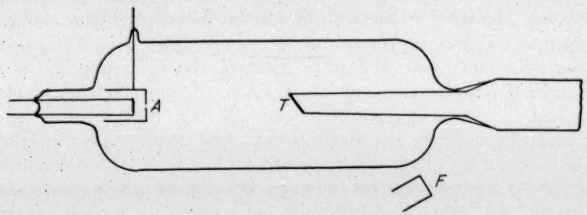
The foregoing statement represents the whole problem. There are, in the last analysis, no light quanta, no X-rays, etc., except in so far as it may be convenient to introduce these concepts as an intermediary scaffolding in the process of the analysis. The only thing that anyone ever *measures* is the presence of such an entity as an electron, or a molecule, within some specified limits of the coördinates. For example, we observe the entry of an electron into a Faraday cylinder, or the darkening of a photographic plate which we associate with the arrival of certain atoms at the plate. Even in the case where the darkening of the plate is produced by what the physicist would term a beam of light, we may, and must, ignore the consideration of the light from the true wave mechanics standpoint, and recognize instead, the chemical changes produced in the silver salt as symbolized by the growth from practically zero to a finite value of the probability of certain associations of values of the coördinates identified with the silver salt. Strictly speaking, a more refined logical sensitivity would drive us still further into the realms of the abstract. We should not leave the electron at the stage where it was endowed with a large probability of being in the Faraday cylinder, we should trace, via wave mechanics, the theory of the Faraday cylinder, of the electrometer to which it is connected, of the spot of light on the scale, and we should even trace the matter through to our brain, and represent in true wave mechanics fashion, the act of writing the ultimate result down in our notebook. In this, we should encounter much complexity, and should have to pause on the way to settle certain subtle questions concerned with the wave mechanics theory of vital processes. We decide there-

fore, not to be too pedantic, but to leave the wave mechanics story of our electron, for example, when it has been calculated as having a large probability of being found in the Faraday cylinder; and, stifling our conscience in the matter of its not having appealed to any of our senses in that position, we are content to carry on the remainder of the story by old classical methods, and take the indication of the electrometer as a criterion for the presence or absence of the electron in the cylinder. In the recognition of all the things which we really do observe, the deflection of the electrometer needle, the darkening of the photographic plate, etc., there becomes involved an association between the actual phenomena observed, and the probability of occurrence of certain of the coördinates of our system, within certain specified limits, an association which permits us to complete the story by classical, physical or chemical means in those parts in which we have no doubt as to the approximate applicability of those means.

It is worthy of notice that, in the spirit of the foregoing attitude towards wave mechanics, there remain none of those difficulties concerned with the propagation of waves in a multidimensional space, difficulties which really do not belong to the subject at all, and which find their origin in an attempt to retain the idea of a wave propagation in a medium—that old medium, the æther—invented by man for the purpose of propagating his misconceptions from one place to another—the æther which of all the subtle fluids invented for the stimulation of the imagination, is the only one which has so far not been prohibited.

In order to illustrate the significance of the foregoing remarks, let us suppose that we are presented with a problem, which in the language of the experimental physicist, would be described as follows: A beam of 100,000 volt electrons comes through the hole A, and strikes the target T, producing X-rays. Some of the X-rays fall upon the plate P from which they eject electrons, some of which are caught in the Faraday cylinder F. We may further complicate the experiment in imagination by supposing that the heating of the plate P by the X-rays is so great that some of the atoms evaporate from the plate and condense on the X-ray tube. It is desired to inquire of wave mechanics as to the story of

the influx of electrons to the Faraday cylinder, and the evaporation of the substance of the plate P on to the glass of the X-ray tube.



Our first step must be to write down a differential equation for ψ. In the stage to which wave mechanics seems so far to have crystallized itself in the matter of writing down this equation the plan is to seek the Hamiltonian function H which would be appropriate to the discussion of the electrons, X-ray target, plate P, etc., as one system on the basis of the old mechanics and then after replacing any momentum p_r by $\frac{h}{2\pi i}\frac{\partial}{\partial q_r}$, treat the resulting expression for H as an operator which operates on ψ, and write for our differential equation

$$-\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_1}, \frac{h}{2\pi i}\frac{\partial}{\partial q_2}, \cdots q_1, q_2, \cdots t\right)\psi.$$

We shall, for a moment, defer any criticism of the significance of this procedure, and rest our attention on the supposition that we have *some* differential equation for ψ. It is now our business to solve it subject to specified initial conditions. Now here we encounter a logical difficulty. Unless we know the form of our differential equation, we are unable to specify the elements which must be assigned in order to determine its solution. Our physical experience tells us that, as regards the conditions at A, at any rate, that which ought to be assigned must provide the equivalent of the assignment

of the density and the velocity of the electron stream coming out of the hole A. However, we have no guarantee, without specification of the equation, that the equation itself will be content with this. Waiving a logically complete answer to this difficulty, we may suppose that at $t = 0$, for example, ψ must assume in the vicinity of the target, of the plate P and of the Faraday cylinder, etc., the values appropriate to some normal states of the atoms contained in those elements of the system, while in the vicinity of A, the initial value must be such that the resulting value of $\psi\bar{\psi}$ in that vicinity will have a finite value over only a limited region, and a value of such amount that the integrated value of $\psi\bar{\psi}$ corresponding to the probability that an electron shall be in that limited region shall have value which corresponds to a definitely assigned amount of electricity in the electron beam which may be supposed to have just emerged from A. This is not all, however, for we have to introduce the idea that the electrons have the assigned velocity of, say 100,000 volts. Presumably we must do this by analyzing our ψ in the neighborhood of the hole A into plane waves, and then requiring of it that it shall consist of such an admixture of plane waves, that if we associate with anyone of them a vector quantity p, defined by $\lambda = h/p$, the amplitude of the wave of length λ shall be such that if we make the probability of momenta lying between p and $p + dp$ depend upon its square, the distribution of momenta p among the electrons will be such as to give a large enough value to the probability of the momentum p corresponding to the velocity 100,000 volts to satisfy our experimentalist that our mathematical representation of the problem corresponds to his facts. The initial conditions of the problem having been expressed, the remainder of the story is to be told by the solution of the differential equation. It must result that the distribution of ψ in the vicinity of the target will remain sensibly what it has always been until such time as is represented by the time of arrival of the electrons there, when it should begin to change rapidly. The experimentalist will want to talk about the emission of X-rays. However, we must not listen to him, but must let ψ tell its own story by gradually propagating its alterations until the regions at P begin to experience such

alterations. The time rates of change of $\psi\bar{\psi}$ should then contain in itself the story of the passage of the electrons from the plate and of their appearance in the Faraday cylinder. It must tell this by the integrated value of $\psi\bar{\psi}$ with respect to all the coördinates, other than those which were inserted in our original differential equation to represent the electrons in the beam, becoming finite at points within the cylinder, whereas, at $t = 0$, and for some time subsequently, the integrated value of $\psi\bar{\psi}$ in question was zero at all points within the cylinder. The story of the evaporation of the metal from the plate A must be told in similar manner by the integrated value of $\psi\bar{\psi}$ with regard to all coördinates other than those introduced to represent the atoms on the plate P becoming finite at points in the vicinity of the glass whereas at $t = 0$ and for sometime subsequently, it was zero.

If the experimentalist should place a stopping potential V on the Faraday cylinder and should seek to measure the velocities of the electrons by adjusting V so as to just keep them out, the wave mechanics story must be that of including V in the original differential equation, and subsequently determining for it a limiting value which just insures that the integrated value of $\psi\bar{\psi}$ referred to above is zero for all time for points inside the Faraday cylinder. If, in the experiment, the glass tube of the X-ray tube were replaced by lead, the lead would also figure in the Hamiltonian function, and in such a way as to cause the solution for ψ outside of the tube to be sensibly independent of all boundary conditions expressed for the regions inside the tube. In this manner the story of the shielding action of the lead would be told.

Now, I am perfectly aware of the fact that the procedure usually adopted in wave mechanics problems is not that outlined above. The procedure actually adopted in such problems is to consider the quantum transitions created in the atoms of the X-ray target by the arrival of the electrons, and then, by treating $\psi\bar{\psi}$, or something closely allied to it, as density of electric charge, determine by its oscillations as produced by a suitably defined velocity associated with it, an electromagnetic wave which falls upon the plate P. The electromagnetic wave then exerts its influence, mathematically, by contributing, through its scalar and vector potentials,

to the Hamiltonian function of the atoms in P, and in this way determines the subsequent probabilities of transitions associated with the photoelectric effect. This procedure, involving as it does, the association of $\psi\bar{\psi}$, or something like it, with charge density, involves ideas not necessarily contained in the more fundamental method of attack; and, if it is to be right, it remains to show that the two procedures are the analytical equivalents of each other. It is no excuse to say that the more fundamental method is too difficult to carry out and that, therefore, we are permitted to adopt a simpler method.

Of course, in the more fundamental method, it is obvious that those features of the electromagnetic theory which provide for propagation with velocity c must, in some way, be provided for in the fundamental operator H, or at any rate in the differential equation which governs ψ. Insofar as classical procedure provides any precedent, the suggestion is to start with a Lagrangian function

$$L = L(q_1, q_2, \cdots \dot{q}_1, \dot{q}_2, \cdots t) \tag{1}$$

and build up a Hamiltonian

$$H = \sum p_r \dot{q}_r - L,$$

expressed, however, as a function of the p's and q's, with p_r defined as

$$p_r \equiv \frac{\partial L}{\partial \dot{q}_r}. \tag{2}$$

Unfortunately, even classical electrodynamic theory does not reveal that part of its content which is concerned with the finite rate of propagation of effects by starting with a Lagragian function as simple as (1). There occur the vector and scalar potentials, which cannot be expressed in terms of instantaneous velocities and positions; and, indeed, the approximation to such a representation robs the whole scheme of the properties of finite rate of propagation. If, however, the Lagrangian function is to contain higher time derivatives than the first, even should we arbitrarily define a set of momenta by relations of the type (2), there remains the impossibility of defining along the conventional lines of

procedure a Hamiltonian function H, as a function of the p's, q's and t, to form a basis for constructing a Schrödinger equation by the replacement of p_r by $\frac{h}{2\pi i}\frac{\partial}{\partial q_r}$.[1] It would seem that we are here confronted with a situation analogous to that prevailing before Maxwell introduced the displacement current into electromagnetic theory; and, just as the attempt to retain the word momentum in the specification of the phenomena necessitated a wider concept of that quantity than could be expressed in terms of the motion of electricity directly, a concept in which the field played a non-eliminatable part, so in wave mechanics it would seem that we must ultimately come to a way of formulating the fundamental equation for ψ which is less naïve, and more widely embracing than that exemplified in the well-known equation

$$-\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_1}, \frac{h}{2\pi i}\frac{\partial}{\partial q_2}, \cdots q_1, q_2, \cdots t\right)\psi.$$

Finally, I would close with this observation. We say that a theory such as the electromagnetic theory is complete when, given unlimited mathematical ability, at any rate, there exists a method of solving any problem in its field when that problem is completely stated, i.e., when the necessary boundary conditions are given, I would suggest that, in this sense, a theory of wave mechanics does not exist today.

[1] At best, we should obtain an equation of high, and possibly infinite order. Concerning such equations there is doubt as to whether they say very much. Thus, confining oneself, for illustration, to non-partial differential equations in t, an equation of the 100th order requires, to complete its solution, the specification of the time derivatives up to the 99th order at $t = 0$; and as we proceed to the limit, the numbers of the derivatives to be expressed tends to infinity. If, however, we express all the time derivatives at $t = 0$, the function is completely determined by McLaurin's expansion, without the differential equation, over the range in which it is analytic; and, if it ceases to be analytic, the whole mathematical procedure requires revision in any case. Speaking rather crudely, we may say that the higher the order of the differential equation, the less it says; and, while in certain cases where the coefficients of the higher derivatives are small, these considerations may be trivial, they may mount to fundamental importance when we come to force out of our expression for ψ the anticipation of events which may be long delayed in such a sense, for example, as that concerned in our imaginary experiment cited earlier, where, long before the arrival of the electrons at P, the ψ function anticipates their arrival, in its structure.

THE OPERATOR H IN SCHRODINGER'S EQUATION.

BY

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IN the short time at my disposal I wish to say a few words about the nature of the operator H which occurs in Schrödinger's equation

$$\frac{ih}{2\pi}\frac{\partial\psi}{\partial t} = H(\psi)$$

and which dominates the wave-theory. The accepted point of view is to proceed on the particle theory of matter setting up the expression for the energy of the system and obtaining the Hamilton-Jacobi partial differential equation. If

$$L = T - V$$

is the kinetic potential and if

$$F = Lt',$$

where primes denote differentiation with respect to an arbitrary independent variable θ of which the coördinates q of the system and t are functions, then

$$H = -\frac{\partial F}{\partial t'}.$$

(If

$$T = \frac{1}{2}g_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta} = \frac{\frac{1}{2}g_{\alpha\beta}q^{\alpha'}q^{\beta'}}{t'^2}$$

then

$$F = \frac{\frac{1}{2}g_{\alpha\beta}q^{\alpha'}q^{\beta'}}{t'} - Vt'$$

and

$$H = -\frac{\partial F}{\partial t'} = \frac{1}{2}\frac{g_{\alpha\beta}q^{\alpha'}q^{\beta'}}{t'^2} + V = \frac{1}{2}\,g_{\alpha\beta}\dot{q}_{\alpha}\dot{q}_{\beta} + = VT + V).$$

The Hamilton-Jacobi equation is obtained by eliminating the $\dot{q}^r$ from the equations

$$p^r = \frac{\partial L}{\partial \dot{q}^r}; \qquad H = T + V.$$

If, then, we wish to construct a wholehearted wave theory we must find a function F which is homogeneous of the first degree in the $n + 1$ derivatives q', t' (n being the number of degrees of freedom of the system); the $n + 1$ derivatives $\frac{\partial F}{\partial q'}$, $\frac{\partial F}{\partial t'}$ are, then, homogeneous functions of degree zero in the derivatives q', t' or, in other words, they are functions of the n ratios of these derivatives. Eliminating these n ratios from the $(n + 1)$ functions we get a relation

$$\varphi\left(\frac{\partial F}{\partial q'}, \frac{\partial F}{\partial t'}\right) = 0.$$

The Hamilton-Jacobi equation follows on setting

$$\frac{\partial F}{\partial q'} = \frac{\partial S}{\partial q}, \qquad \frac{\partial F}{\partial t'} = \frac{\partial S}{\partial t}$$

where S is a function of (q, t) obtained by integrating F from a fixed point along an extremal curve for the integral $\int F d\theta$. The wave equation follows by considering those wave equations which have the surfaces S = constant as wave fronts. The problem is, therefore, essentially that known as the inverse problem of the Calculus of Variations; the surfaces S being regarded as of physical significance one must start with them and work back to F.

In conclusion we would like to point out that there are certain advantages in general dynamical theory in the introduction of a state-energy space of $2n + 2$ dimensions instead of the usual state-space of $2n + 1$ dimensions. In this state-energy space the coördinates are (q, t, p, H) and a spread of $2n + 1$ dimensions $\phi(q, t, p, H) = 0$ defines a dynamical system. The dynamical theory of contact transformations becomes identical with Lie's theory of geometrical contact transformations and the symmetry given to the situation seems to confer a real simplicity.

QUANTUM MECHANICS OF MOMENTUM SPACE.

BY

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IN Schrödinger's work on quantum mechanics he starts from the Hamiltonian function of the mechanical system under investigation and makes from it an operator by the process of replacing each momentum, p_σ, by the differentiation operator $\frac{h}{2\pi i}\frac{\partial}{\partial q_\sigma}$. A wave function $\psi(q_1 \cdots q_n)$ is then introduced which is required to satisfy the equation

$$H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right)\psi = W\psi, \tag{1}$$

where W is a constant. That is, the energy operator, operating on ψ must give simply a constant into the same function. Only such ψ's as are finite and continuous throughout the range of the q's are allowed. It is then found that such ψ's exist only when W has certain characteristic values depending on the form of the operator H. These values of W are then taken as the allowed values of the energy. The interpretation given by Schrödinger for his function ψ however is not tenable and must be given up in favor of one due to Born that the quantity $\psi(W, q)$ is related to the probability of finding the system in a little volume element of the coördinate space, dq, when it is known that the system is in the energy state W. This probability is in fact taken equal to

$$\psi(W, q)\overline{\psi}(W, q)dq, \tag{2}$$

where $\overline{\psi}$ is the conjugate complex function to ψ. This method of finding energy levels and this interpretation of ψ are at the heart of many of the successes of the new quantum mechanics.

As Professor Slater has pointed out, this statistical feature runs throughout the whole of quantum mechanics. The

questions which it attempts to answer are to be phrased in the following form: (*A*) What are the numerical values which a physical quantity may assume, and (*B*), In a given set of circumstances what is the relative probability that it does assume any one of these allowed values? In terms of this general formulation one sees that the equation of Schrödinger (1) and the expression (2) form answers to two very special questions of this type. Equation (1) is the means employed for finding what values the particular physical quantity, total energy of the system, may have. It answers a question of the first type. Expression (2) is the answer to a question of the second type. We are supposed to know that the configuration may assume any value represented by the allowed ranges of the q's. The given set of circumstances is the certain information that the system is in the state of energy W and then the expression gives relative probabilities that the system be found in various elements of the configuration space.

It has been the service of Dirac and Jordan to point the way to the generalization of equations (1) and (2) so that the means is at hand to answer any questions of the type (*A*) and (*B*) rather than simply the special cases of them covered by the Schrödinger equation. The purpose of these remarks is to illustrate the ideas by the presentation of a few simple cases in which one works with the momenta as independent variables, i.e. in which an equation analogous to (1) is utilized to find a $\psi(W, p)$ which can be used in an expression analogous to (2) to get the relative probability that the momenta will lie in the element of momentum space defined by p_1 between the values p_1 and $p_1 + dp_1$, p_2 between the values p_2 and $p_2 + dp_2$ and so on. It is believed that the results, while containing nothing new of importance to atomic physics will nevertheless be useful in facilitating the study of the general theory.

To each physical quantity can be associated an operator and the laws of association of operators with quantities constitute an important branch of the subject. There are different representations of a physical quantity and the letter denoting the quantity has to be used in two senses. Where written in bold type it will be meant as the quantity in its

capacity as an element of the algebra of quantum mechanics. That is, as a matrix in many cases, or more generally as what Dirac calls the q-number ($q \sim$ quantum). In the language of Weyl's recent book,[1] each physical quantity is associated with an operator in the unitary function space. Where ordinary type is used the ordinary real variable which is the arithmetic value of the quantity is meant. The guiding principle in associating operators with the physical quantities is that this must be done consistently with the matrix mechanics of Heisenberg. Thus between coördinate and conjugate momentum one has the rule

$$\mathbf{pq} - \mathbf{qp} = \frac{h}{2\pi i}. \tag{3}$$

In getting Schrödinger's equation (1) we associate each $\mathbf{q}$ with the operation: Multiplication by the arithmetic value q and each $\mathbf{p}$ with the operation: Multiplication by $h/2\pi i$ and partial differentiation with respect to the conjugate q. One sees that this is in fact consistent with (3) since

$$\frac{h}{2\pi i}\frac{\partial}{\partial q}(q\psi) - q\frac{h}{2\pi i}\frac{\partial \psi}{\partial q} \equiv \frac{h}{2\pi i}\psi. \tag{3a}$$

But another way[2] of associating operators with $\mathbf{p}$ and $\mathbf{q}$ which is just as consistent with (3) as the method leading to Schrödinger's equation is the following:

$\mathbf{p}$: Multiplication by p.

$\mathbf{q}$: Operation $-\frac{h}{2\pi i}\frac{\partial}{\partial p}$.

for this gives

$$p\left(-\frac{h}{2\pi i}\frac{\partial \psi}{\partial p}\right) + \frac{h}{2\pi i}\frac{\partial}{\partial p}(\psi p) \equiv \frac{h}{2\pi i}\psi. \tag{3b}$$

so that this association is consistent with (3).

[1] Weyl, "Gruppentheorie und Quantenmechanik." Leipzig, Hirzel, 1928.

[2] This method of introducing the operator $-\frac{h}{2\pi i}\frac{\partial}{p}$ for q when p is multiplication by p was suggested to me after the symposium by my colleague, Dr. H. P. Robertson. It is so neat, that I have taken the liberty of using it rather than the method used in my remarks.

Before passing to the use of this method in determining energy values it is essential to consider the question: What values may each $\mathbf{p}$ assume? Because of the kinematical connection between $\mathbf{p}$ and $\mathbf{q}$ the allowed values of $\mathbf{p}$ are determined by the allowed values of $\mathbf{q}$, as Jordan [3] has shown. As to the allowed values of $\mathbf{q}$ these are supposed to be known almost intuitively from the nature of the system. Thus, the allowed values of a Cartesian coördinate are all real values from $-\infty$ to $+\infty$; of a radius vector in polar coördinates are from 0 to $+\infty$; of the azimuthal angle in spherical or cylindrical coördinates, 0 to 2π; of the co-latitude in spherical coördinates, 0 to π. From these allowed ranges in $\mathbf{q}$ certain results for the allowed values of the associated $\mathbf{p}$ are found. Thus for a Cartesian coördinate or a radius vector whose range is infinite the $\mathbf{p}$ may take on all values from $-\infty$ to $+\infty$. But for an azimuthal angle like φ the allowed values of the associated $\mathbf{p}$ are the discrete range given by $n(h/2\pi)$ where n takes on all *integral* values from $-\infty$ to $+\infty$.

Corresponding to this second mode of associating operators with p and q one finds as the equation analogous to (1),

$$H\left(p, -\frac{h}{2\pi i}\frac{\partial}{\partial p}\right)\psi = W\psi, \tag{4}$$

which will now be considered in some simple instances:

The Free Particle.—For the motion of a particle of mass μ in one rectilinear coördinate, x, under no forces the Hamiltonian function is $\mathbf{H} = (1/2\mu)\mathbf{p}^2$. As it is independent of $\mathbf{q}$, (4) takes the simple algebraic form

$$\left(\frac{1}{2\mu}p^2 - W\right)\psi = 0. \tag{4a}$$

It is at once apparent that if p has a value differing from $\pm\sqrt{2\mu W}$ then ψ must be zero. As the range of p is confined to real values one has $W > 0$. But here one strikes a characteristic feature of the theory in that if one adheres to the notion that ψ is to be normalized with respect to the complete allowed range of p, i.e.,

[3] Jordan, Zeitschrift für Physik, **44**, 1 (1927).

$$\int_{-\infty}^{+\infty} \psi(W, p)\bar{\psi}(W, p)dp = 1$$

then ψ must be strongly infinite at $p = \sqrt{2\mu W}$ or $-\sqrt{2\mu W}$ or both in order that the integral shall not vanish. Thus ψ is a function of the type called a δ-function by Dirac:

$$\delta(x) = 0 \text{ for } x \neq 0 \quad \text{and} \int_{-\infty}^{+\infty} \delta(x)dx = 1.$$

Plainly one does not deal here with a simple requirement of finiteness and continuity for $\psi(W, p)$ as seems usually to be the case with the $\psi(W, q)$ of Schrödinger.

The analogue of (2) is that this $\psi(W, p)\cdot\bar{\psi}(W, p)dp$ is equal to the probability that p have a value between p and $p + dp$ when W is known to have the value W. The equation (4a) shows that this vanishes except for

$$\frac{1}{2\mu}p^2 = W,$$

so that we recognize in this an equation of classical mechanics which retains its validity in quantum mechanics.

All values of W are on an equal footing in regard to the nature of the $\psi(W, p)$ associated with them so (4a) would show that all values of W greater than zero are allowed.

The Freely Falling Particle.—This is the same as the preceding except that now the Hamiltonian is $(1/2\mu)\mathbf{p}^2 - \kappa\mathbf{x}$ so (4) becomes

$$\left(\frac{1}{2\mu}p^2 + \frac{\kappa h}{2\pi i}\frac{\partial}{\partial p}\right)\psi = W\psi \tag{4b}$$

of which the solution is

$$\psi = \exp.\frac{2\pi i}{\kappa h}\left(Wp - \frac{1}{6\mu}p^3\right).$$

This solution is finite throughout the range of values of p no matter what the value of W, therefore all values of W are allowed, both positive and negative, agreeing with the classical mechanics of uniformly accelerated motion. The value of $\psi\bar{\psi}$ is here independent of p which means that all

values of p are equally probable. This is in accord with the classical motion if we understand that all values of the time are to be taken as equally probable in reckoning the relative probabilities for the momentum. That this is the correct analogue with classical mechanics is consistent with quantum mechanics for exact specification of the energy W, makes the conjugate quantity, the time, to be wholly indeterminate, as is well known.[4]

The Harmonic Oscillator.—Here the Hamiltonian is

$$\frac{1}{2\mu}\mathbf{p}^2 + \frac{1}{2}\kappa\mathbf{x}^2$$

and because it is quadratic in both p and x one gets an equation of the same form here (from (4)) as by Schrödinger's method, (1). It is:

$$-\frac{h^2\kappa}{8\pi^2}\frac{\partial^2\psi}{\partial p^2} + \frac{1}{2\mu}p^2\psi = W\psi. \tag{4c}$$

Requiring that ψ be finite at $p = \pm\infty$ this gives the energy levels

$$W_n = \left(n + \frac{1}{2}\right)h\nu, \qquad n = 0, 1, 2 \cdots, \qquad \nu = \frac{1}{2\pi}\sqrt{\frac{\kappa}{\mu}};$$

and

$$\psi_n(p) = \exp.\ (-p^2/p_0^2)H_n(p/p_0), \qquad p_0^2 = \frac{h}{2\pi}\sqrt{\kappa\mu}.$$

Here $H_n(x)$ is the nth Hermitian polynomial as with Schrödinger. One sees that p_0 is the maximum value of the momentum in classical oscillations of energy $\frac{1}{2}h\nu$ just as the characteristic length, a, occurring in Schrödinger's functions was the classical amplitude of oscillation in the classical motion of this energy. For various values of n therefore these ψ functions bear the same relation to the momentum values associated with the classical motions of the same energy as they do with Schrödinger when they give the coördinate probabilities instead of momentum probabilities.

[4] Heisenberg, Zeitschrift für Physik, **43**, 172 (1927) and Bohr, Nature, April 14, 1928.

Free Rotator with Fixed Axis.—The Hamiltonian is $(1/2\mu)\mathbf{p}^2$ as with the free particle moving in a straight line. Here μ is the moment of inertia. But as the coördinate now is one restricted to values from 0 to 2π, it follows that p is restricted to values $n(h/2\pi)$ with n integral. Therefore the allowed values of W are

$$W_n = \frac{n^2h^2}{8\pi^2\mu}$$

with ψ functions of the same type as with the free particle.

The Physical Pendulum.[5]—This offers an interesting variation of the preceding, the Hamiltonian being $(1/2\mu)\mathbf{p}^2 - \kappa \cos \theta$. To form equation (4) calls for interpretation of the operator $\cos \theta$. This is clearly

$$\cos\frac{h}{2\pi i}\frac{\partial}{\partial p} = \frac{1}{2}\left[\exp.\left(\frac{h}{2\pi i}\frac{\partial}{\partial p}\right) + \exp.\left(-\frac{h}{2\pi i}\frac{\partial}{\partial p}\right)\right].$$

Recalling that the terms on the right are a symbolic form of Taylor's theorem it is plain that the operation on $\psi(p)$ gives

$$\cos\theta\psi(p) = \frac{1}{2}\left[\psi\left(p + \frac{h}{2\pi}\right) + \psi\left(p - \frac{h}{2\pi}\right)\right]$$

which remains consistent with the general result that the allowed values of p differ by $h/(2\pi)$ for the momentum conjugate to the coördinate θ of period 2π. Introducing $p = n(h/2\pi)$ the equation (4) for $\psi(W, n)$ becomes

$$\frac{\kappa}{2}[\psi(n+1) + \psi(n-1)] = \left(W - \frac{n^2h^2}{8\pi^2\mu}\right)\psi(n) \quad (4d)$$

The allowed energy levels for W are those for which this difference equation for ψ has solutions which remain finite as n takes on integral values between $-\infty$ and $+\infty$. But there seems not to be a very useful mathematical literature on boundary value problems for difference equations.

In these examples we see cases where the equation (4), which is analogous to (1), in momentum space may be alge-

[5] For the treatment of this problem by Schrödinger's method see Condon, Physical Review, 31, 891 (1928).

braic or may be a finite difference equation. In coördinate space, since the Hamiltonian always includes a quadratic form in the momenta a second order differential operator is always generated. This makes (1) always resemble the partial differential equations of wave propagation from elasticity theory. In fact this analogy is fundamental in Schrödinger's approach to quantum mechanics. From the more general standpoint this resemblance to a wave equation is somewhat accidental.

PALMER PHYSICAL LABORATORY,
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THE STATISTICAL INTERPRETATION OF VARIOUS FORMULATIONS OF QUANTUM MECHANICS.

BY

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THE various formulations of the quantum mechanics, although to a large extent fundamentally equivalent, are so numerous and varied in superficial structure that they are apt to confuse the casual student. Some of the more important mathematical forms are the following:

1. Born and Heisenberg's matrix theory.[1]
2. Schroedinger's wave mechanics.[2]
3. Dirac's theory of q-numbers.[3]
4. Born and Wiener's operator theory.[4]
5. The transformation theory of Dirac, Jordan, and Pauli.[5]

Numbers 1 and 2 are the commonest, and those which will be discussed primarily in this paper. The fifth is, however, the most comprehensive and far reaching, as it includes the others as special cases. The formulation of the quantum mechanics given by Professor Weyl in the first part of his recent book[6] can to a great extent be regarded as a very beautiful geometrical interpretation of the transformation theory in the Hilbert space, and stands in roughly the same relation to this as Minkowski's four dimensional world to special relativity or non-Euclidean geometry to general

[1] Born, Heisenberg, and Jordan, *Zeits. f. Physik*, **34**, 858 and **35**, 557 (1925–26).

[2] E. Schroedinger, "Abhandlungen zur Wellenmechanik" (Leipzig, 1927; English translation, London, 1928).

[3] P. A. M. Dirac, *Proc. Roy. Soc.*, **109A**, 642; **110A**, 561; **111A**, 281 (1925–6).

[4] Born and Wiener, *Jour. of Math. and Phys., Mass. Inst. of Tech.*, **5**, 84 (1926) or *Zeits. f. Physik*, **36**, 174 (1926).

[5] P. A. M. Dirac, *Proc. Roy. Soc.*, **113A**, 621 (1927); Jordan, *Zeits f. Physik*, **40**, 809, **41**, 797, **44**, 1 (1926–7); (cf. also London, *Zeits. f. Phys.*, **40**, 193 (1926)).

[6] H. Weyl, "Gruppentheorie und Quantenmechanik" (Leipzig, 1928).

relativity. The appeal depends largely on whether one is geometrically or analytically inclined.

The *mathematical* relations of equivalence between these various formulations are fairly well known. What I wish to emphasize is how the statistical interpretation of quantum mechanics, which has been so nicely described in Professor Slater's paper in this symposium, serves as a focal *physical* viewpoint and unifying influence which blends together these different formulations and makes them no longer seem distinct mathematical entities. There is one fundamental physical lemma which epitomizes the spirit of the whole statistical outlook, just as the invariance of the velocity of light does the spirit of relativity or the second law that "heat doesn't flow uphill" does the spirit of thermodynamics. This lemma is "Heisenberg's indeterminism principle." [7]

Heisenberg's Indeterminism Principle.—"It is impossible to specify or measure accurately and simultaneously both a coördinate q and its conjugate momentum p." The experiment itself inherently spoils the possibility of such a simultaneous determination. Instead the product of the errors Δq, Δp in determining q and p respectively is of the order of Planck's h, so that

$$\Delta q \Delta p \sim h.$$

An important special case is where q is a Cartesian coördinate; then $p = mv$ and consequently it is impossible to specify simultaneously the position and velocity of an electron. Another important special case is where the p's are "action variables" J_i which take on quantized values $n_i h$ and which thus specify the stationary state or energy level. In this instance the conjugate coördinates are phase angles ("angle variables"). Thus when we specify a stationary state we cannot specify the phase of an electron in its orbit. We shall soon see that this item furnishes the clue to the physical interpretation of the Heisenberg matrices.

A situation which is of particular importance for our uses is that which arises when one variable of a canonical pair is specified with complete precision, an ideal state which is

[7] W. Heisenberg, *Zeits. f. Physik*, **43**, 172 (1927); N. Bohr, *Nature*, **121**, 580 (1928).

perhaps never reached completely but which nevertheless is fulfilled to all intents and purposes. If q be thus accurately specified, there is an infinite error in its conjugate momentum p. More precisely, this means that all values are equally probable for p. All we can then measure about a function $f(p, q)$ is its average with respect to p, which in classical theory would be proportional to the expression $\int f(p, q)dp$.

Various simple illustrations of the indeterminism principle, such as the "gamma ray microscope" are already becoming so hackneyed that I shall not attempt to repeat them. At first thought this principle appears contrary to our experience. The main problem, for instance, of driving an automobile is that it certainly does have simultaneously both position and velocity. However, such "large scale" phenomena, unlike occurrences inside the atom, involve such large amounts of action compared to Planck's h, that the limits of precision fixed by the indeterminism principle become negligible.

We must now apply the indeterminism principle to the interpretation of the various formulations of the quantum mechanics.

Matrix Theory.—Historically the first form of the quantum mechanics to make its appearance was the matrix theory. The indices of the Heisenberg matrices refer to definite stationary states or energy levels. By the indeterminism principle the phase or epoch angles are indeterminate as soon as we specify the quantum numbers. Hence the only observable physical quantities belonging to a given stationary state are those which are averages over all phases or positions of the electron in its orbit, and so we cannot localize an electron at a definite position at a definite time. Thus one can no longer speak of an aphelion or perihelion distance in the atomic orbit. One has, in fact, no way of measuring these distances experimentally, and it is the essence of the philosophy of the new mechanics that it should deal with only observable quantities. One can, however, still retain the concept of a mean square radius. This, of course, cannot be determined experimentally by ordinary means of measuring distances such as meter sticks, but indirectly is really observable. It can be shown, for instance, that the diamagnetic susceptibility is proportional to this mean value of r^2, which

is a diagonal Heisenberg matrix element $r^2(n;\, n)$. In general, *the diagonal elements of a Heisenberg matrix representing a function f give the average value of f when the quantum numbers are specified and all phases are equally probable.*

As to the non-diagonal elements, it is my opinion that they do not have any direct physical reality, but are only mathematical aids ("Hilfsmittel") to the calculation of the diagonal elements of other matrices. Thus as by the matrix law of multiplication $r^2(n;\, n)$ equals $\Sigma_k r(nk)r(kn)$, it is evident that in order to know the diagonal elements of r^2, it is not sufficient to know the diagonal elements $r(nn)$ of the first power of r, but we must also know the off-diagonal elements $r(nk)$. One may urge that in the existing quantum theory a certain amount of reality is given to the non-diagonal elements by interpreting them as transition probabilities, and it is true that the proper formulas for the rate of radiation can be obtained by substituting the off-diagonal matrix elements for the Fourier coefficients in the classical radiation formulas. Such a procedure, however, is only an extrapolation from the classical theory, like the correspondence principle in the old Bohr quantum theory, rather than something self-contained in the quantum mechanics. When we are really able to calculate from purely quantum considerations the rate of radiation we will doubtless find that these off-diagonal elements in a conservative system are important because they give the average rate or diagonal term of the radiation when the system is made non-conservative by considering external radiation fields and also the force on an electron due to its own radiation. Calculations by Dirac, Slater and Born [8] have already verified this for absorption and forced emission, but our theory of spontaneous emission is at present less complete. The law of conservation of energy in the statistical viewpoint means that there is no difference between the mean square and the square of the mean of the energy function H, and from this one readily sees with the matrix law of multiplication that H must be a diagonal matrix.

Wave Mechanics.—We must now turn to the Schroe-

[8] P. A. M. Dirac, *Proc. Roy. Soc.*, **112A**, 661; **114A**, 243 (1926–7); Slater, *Proc. Nat. Acad. Sci.*, **13**, 7 and 104 (1927); M. Born, *Zeits. f. Phys.*, **40**, 167 (1926).

dinger wave mechanics, which is probably the most popular and simplest way of presenting elementary quantum theory. Here the all important question is the physical interpretation of the wave function ψ, and in this one must steer between two extremes. One extreme is to say that the electron is a fluid-like substance with a density proportional to $|\psi|^2$. In this way one can make considerable progress towards constructing what I may term a "hydrodynamical" model of the atom which retains a simultaneously determinate time and position. This idea appeals to many physicists who are rather loath to accept the revolutionary philosophy of Heisenberg's indeterminism principle. It is, I believe, the opinion of the majority of theoretical physicists, including my own, that this hydrodynamical, non-statistical viewpoint is not tenable when pushed too far. I shall mention only three difficulties: first, the hydrodynamical theory is rather difficult to extend to a system with more than one electron because more than three dimensions are required; second, point potentials are used in the equations despite a continuous charge distribution, and third, perhaps the most serious difficulty, the rate of spontaneous radiation comes out proportional to the product of the concentrations of electrons in the initial and final states rather than to just the concentration in the initial state.[9] It is to be clearly understood

[9] This argument, which seems to have caused some misunderstanding at the time this paper was presented, may be elaborated as follows: The general Schroedinger equation which has the operator $h\partial\cdots/2\pi i\partial t$ in place of the energy constant W has the solution $\psi = \Sigma c_n\psi_n \exp.\ (2\pi i W_n t/h)$, where the c's are arbitrary constants and $\psi_n(x, y, z)$ is a solution of the restricted Schroedinger equation having the characteristic value W_n. If we regard $|\psi|^2$ as proportional to a fluid charge density, the term which results from the multiplication of $c_n\psi_n \exp.\ (2\pi i W_n t/h)$ and $c_m^*\psi_m^* \exp.\ (-2\pi i W_m t/h)$ in the vector electrical moment $\mathbf{p} = -e\int\int\int \mathbf{r}|\psi|^2 dv$ has a time factor exp. $[2\pi i(W_n - W_m)t/h]$ in accord with the Bohr frequency condition, forming an attractive argument for the "field" view formerly adopted by Gordon, Madelung, Schroedinger, etc. (*Zeits. f. Physik*, **40**, 117 and 322, 1926; *Annalen der Physik*, **81**, 109; **82**, 265, 1926–7). However, such a term in the moment when squared to give the rate of radiation according to the classical formula $2\ddot{\mathbf{p}}^2/3c^3$, will have a factor $|c_n|^2|c_m|^2$. Now the square of the absolute value of a coefficient c_i is ordinarily interpreted as giving the statistical concentration of atoms in the state i. Dirac, Slater, and Born have shown that this interpretation, which is in complete accord with transformation theory, gives just the right value for the Einstein

that these objections are not raised against the proper use of the wave theory, but only against the extreme hydrodynamical interpretations (the so-called "field theories"), now largely abandoned, which do not in some way or other embody the statistical outlook involved in the indeterminism principle.

The other extreme is to regard the wave function as only a mathematical tool for calculating the matrix elements, for it is well known that the Heisenberg matrix elements for any function $f(p_1, \cdots, p_f; q_1, \cdots, q_f)$ are given by the expression [10]

$$f(nm) = \int \cdots \int \psi_n^* f\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_1}, \cdots, \frac{h}{2\pi i}\frac{\partial}{\partial q_f}; q_1, \cdots, q_f\right)\psi_m dq_1 \cdots dq_f,$$

where as usual the asterisk denotes the conjugate imaginary and ψ_n means the normalized wave function for the state n. This is usually by all odds the easiest method of calculating the matrix elements. It is my feeling that in most applications of the quantum mechanics to specific problems, the computer will do best to have a firm grasp of both the matrix and differential equation methods. It is true that the infinite chains of difference equations furnished by the pure matrix theory would be almost impossible to solve without the above Schroedinger-Eckart formula, but it is likewise true that the solution of Schroedinger's equation when there is a small perturbing force is greatly facilitated by using the matrix language. It is my feeling that the latter feature is not properly appreciated by some writers who go through complicated manipulations of a differential equation and think they have found something radically new when all they are really

absorption coefficients. Either one must sacrifice this very satisfactory interpretation of the c's, as does Klein (*Zeits. f. Physik*, **41**, 420, 1927), or else assume that the radiation is proportional to both the initial and final concentrations, which, as we might surmise, is contrary to experiment (see Gaviola, *Nature*, Nov. 17, 1928). An escape from this dilemma is, of course, that the mechanism of radiation is not as simple as supposed in the too naïve hydrodynamical or field theory.

[10] E. Schroedinger, *Annalen der Physik*, **79**, 734 (1926); C. Eckart, *Phys. Rev.*, **28**, 711 (1926); Dirac., ref.[8]

doing is just to sum a few matrix elements often already calculated by others.

The wave function, however, is surely entitled to more physical interpretation than as just a mathematical tool. It is perhaps best described by stating that the square of its absolute value is proportional to what may be termed the "statistical" or average charge density. This can be seen from the above matrix formula as follows. Suppose we take f to be a function which is zero everywhere except in a small element of volume $dv = dq_1 \cdots dq_f$ and which has the value unity there. Then with $n = m$ the integral clearly reduces to

$$\psi_n^* \mathrm{I} \psi_n dv = |\psi_n|^2 dv$$

as only in dv is there a non-vanishing integrand. This expression records the fractional amount of time the electron spends in dv just as a stop watch which is set in operation only when one is in a certain room of a building will, when divided by the total elapsed time, record the fractional amount of time which one spends in that particular room. It may seem rather paradoxical that while one can never say when an electron is in a certain position, one can nevertheless evaluate the fraction of time it is in this configuration.

One may elucidate this concept of a statistical charge density by a rather foolish illustration. If one tried to photograph a firefly on a dark night, it might be necessary to take a time exposure rather than a snapshot. If the firefly is active and flutters all around, the photographic plate would then record a nebulous cloud, and from this a person seeing only the photograph might argue that a firefly is a fluid-like substance pervading all space rather than a discrete bug. It seems to me that the same fallacy is committed in the hydrodynamical model of the electron in wave mechanics. It is the essence of Heisenberg's indeterminism principle that all the information which we can get about the motion of an electron in a stationary state is the time exposure, as when we specify the energy all values of the conjugate variable, namely the time, become equally probable. The electron is not a wave pervading all space, but instead it is a discrete entity or "bug" whose probability of location often shows wave-like maxima and minima. Instead of taking the time

average for one atom, it is, of course, equally legitimate to calculate the instantaneous average over an ensemble of atoms all in the same stationary state. The result will be the same in either case, just as a snapshot of a multitudinous swarm of fireflies will resemble the time exposure for the single fly.

Transformation Theory.—A great service of Dirac and Jordan [5] was to interpret the quantum mechanics as a transformation from one coördinate system to another. Space will not permit me to discuss the mathematics of the transformation theory, but I do want to intimate how it includes the Heisenberg and Schroedinger theories as special cases. In a classical transformation from one set of canonical coördinates α, β to another p, q the values of the new coordinates p, q are determined as soon as we know α, β. In quantum mechanics we cannot specify both α and β simultaneously, but we can nevertheless determine the probability that any function $f(p, q)$ will take on a given value when say α is given accurately and all values of β are supposed equally probable. According to Dirac and Jordan each transformation from one coördinate system to another is characterized by a certain "probability amplitude" $\Phi(q, \alpha)$ such that [11] $|\Phi(q, \alpha)|^2 dq$ is the probability that q will fall in the interval dq when α is given and all values of β are equally probable. The Schroedinger wave function is a special case of such a probability amplitude or transformation function, while the Schroedinger wave equation is a special case of the differential equation for determining the probability amplitude. The specialization which yields the Schroedinger theory is the restriction that the α, β coördinate system be the quantum analog of a set of action and angle variables which make the energy a function only of α, i.e., a diagonal matrix. The Schroedinger wave function can thus be regarded as associated with a transformation from positional coördinates to a set of action and angle variables. It involves two sets of arguments, viz., the positional coördinates q_i and the α_i or quantum numbers, as α_i equals $n_i h$ if it is an action variable. Similarly the Heisenberg matrices are special cases

[11] We restrict ourselves to "real" or "Hermitianized" systems, so that $\Phi(\alpha, q)$ is the conjugate imaginary of $\Phi(q, \alpha)$. See ref.[5]

of the matrices used in transformation theory, as the former are indexed ("labeled") with respect to the quantum numbers or action variables, whereas the latter can be indexed with respect to any set of coördinates α. The most common and important transformations are, of course, those built on the energy function, yielding the Schroedinger and Heisenberg theories, just as in classical theory the most important contact transformations are those connected with the Hamilton-Jacobi equation or with action and angle variables. However, the ability to use more general transformations in quantum theory, corresponding to the same degree of generality as the complete group of contact transformations in classical theory, is occasionally useful as well as philosophically satisfying. Space will not permit a discussion of the group property of these transformations, how they can be regarded as rotations in the Hilbert space, or how probabilities are compounded. In the latter we encounter the rather curious "interference of probabilities" whereby probability amplitudes, i.e., square roots of probabilities are superposed rather than probabilities themselves, but which bears some resemblance to the superposition principles in classical electromagnetic theory as discussed in Professor Wiener's paper.

Asymptotic Connection with Classical Theory.—As already stated by Professor Slater in this symposium, quantum mechanics is an extension not of ordinary mechanics but rather of statistical mechanics. This fact comes to light clearly in the nature of the asymptotic connection between the classical and quantum theories, and most of the balance of the present article will be devoted to showing how the quantum expressions for probabilities and averages merge asymptotically for large quantum numbers into classical expressions if, and only if, the classical mechanics is supplemented by statistical assumptions involved in integrating over certain variables, usually phase angles. To my mind this comparison with classical theory is very instructive in acquainting one with the true physical nature of quantum mechanics.

First let us take a very simple case, a particle of mass m moving in one dimension subject to a potential $V(x)$, and confined to a stationary state of energy W. The wave equation is then

$$h^2(d^2\psi/dx^2) + 8\pi^2 m(W - V)\psi = 0. \tag{1}$$

According to quantum theory the expression

$$|\psi|^2 dx \tag{2}$$

measures the probability that a particle in this state be in the interval dx. What is the analogous classical expression? The variable canonically conjugate to the energy is the time, and we must therefore supplement the straight analytical mechanics by the assumption that all values of the time, i.e., all phases or epochs in a system with just one degree of freedom, are equally probable if the energy is given, so that the probability that a particle be in an interval dx is to be taken proportional to the time dt spent in this interval. The places where it moves slowest are therefore those where it is most likely to be. As by the law of conservation of energy $\frac{1}{2}m\dot{x}^2 + V = W$, the probability of being in dx is therefore

$$Cdt = C\frac{dt}{dx}dx = C\left[\frac{m}{2(W - V)}\right]^{1/2} dx, \tag{3}$$

where C is a constant. We must now show that there is an asymptotic agreement between the expressions (2) and (3). To do this we shall adopt the artifice of solving the Schroedinger equation (1) as a power series in Planck's constant h. The convergence of this procedure should be most satisfactory when dealing with large quantum numbers.[12] Let us change the dependent variable in (1) from ψ to S by the substitution

$$\psi = Ae^{2\pi iS/h} \quad \text{with} \quad S = S_0 + hS_1 + h^2S_2 + \cdots, \tag{4}$$

where A is a constant. Then on removal of the exponential factor after differentiation, Eq. (1) becomes [13]

$$\left[\frac{1}{2m}\left(\frac{\partial S_0}{\partial x}\right)^2 + V(x) - W\right] + h\left[\frac{1}{4\pi i}\frac{\partial^2 S_0}{\partial x^2} + \frac{\partial S_0}{\partial x}\frac{\partial S_1}{\partial x}\right] + h^2[\cdots] + \cdots = 0. \tag{5}$$

[12] The convergence of this development is open to grave question, though the series nevertheless seem to give significant results at least for the first two terms. Cf. H. A. Kramers, ref.,[18] also C. Eckart, *Zeits. f. Physik*, **48**, 295 (1928).

[13] We do not develop W as a power series in h, as the necessity for this can be avoided without loss of generality by comparing the quantum system with a classical one of exactly the same energy. See ref.[20]

The values of S_0, S_1, S_2 $\cdots$ are determined by equating to zero the terms in (5) which are respectively constant and proportional to h, h^2, $\cdots$, etc. The differential equations so obtained for S_0 and S_1 are immediately integrable, yielding

$$S_0 = \int [2m(W - V)]^{1/2} dx;$$

$$4\pi i S_1 = -\log (\partial S_0/\partial x) = -\log [2m(W - V)]^{1/2}. \quad (6)$$

One immediately recognizes S_0 as the classical action function, and the differential equation by which it is determined, viz., the part of (5) which does not disappear in the limit $h = 0$, is the classical Hamilton-Jacobi equation. Hence the logarithm of the Schroedinger wave function is by (4) roughly proportional to the classical action function. The requirement that the wave function be single valued means that the exponent in (4) must increase by an integral multiple n of $2\pi i$ when the radical in the integrand of (6) traverses a complete cycle of values. In the first approximation, where $S = S_0$, this means that

$$\oint [2m(W - V)]^{1/2} dx = \oint p dx = nh,$$

which is nothing other than the Sommerfeld-Wilson quantum condition. Hence we see why the old Bohr theory was a good approximation in many cases.

This derivation of the Sommerfeld-Wilson quantum conditions as a first approximation by means of the substitution (4) is due to Wentzel and Brillouin.[14] To obtain the asymptotic identity of the classical and quantum probabilities it is, however, necessary to go a step further to the second approximation S_1. We must now distinguish between two regions I and II of values of x:

I. One region is that where the radical $[W - V]^{1/2}$ is real. This is the region in which the potential energy is less than the total energy, and is therefore the interval inclosed between the classical libration or oscillation limits. Here S_0 is real and the absolute value of the factor $e^{2\pi i S_0/h}$ is unity. Hence by (4) and (6)

$$|\psi|^2 dx = A^2 |e^{2\pi i S_1}|^2 dx = [A^4/2m(W - V)]^{1/2} dx. \quad (7)$$

[14] G. Wentzel, *Zeits. f. Physik*, **38**, 518 (1926); L. Brillouin, *Jour. de Physique*, **7**, 353 (1926).

This is the same as (3), thus giving the desired asymptotic agreement between the classical and quantum results. The constants of proportionality $Cm^{1/2}$ and $A^2/m^{1/2}$ in (3) and (7) respectively, which would otherwise be arbitrary, become equal when account is taken of the normalization, both classical and quantum, that the total integrated probability $\int_0^{\tau/2} Cdt$ or $\int_{-\infty}^{\infty} |\psi|^2 dx$ must equal unity.[15]

II. Let us now turn to the other region in which the radical and S_0 are imaginary, and which is exterior to the classical region of motion as it would demand more potential energy than the total energy W. Here the exponent in $e^{2\pi i S_0/h}$ is a positive or negative real number which increases monotonely in absolute value as we go further and further away from the classical libration limits. If the exponent is positive the wave function increases without limit and is not a characteristic function ("Eigenfunktion") of the differential equation. If it is negative the wave function will by (4) drop off in a rapid exponential fashion beyond the classical libration limits, and so the statistical density can be considered as virtually zero outside these limits. In general only certain particular values of the energy constant W may give this rapid diminution instead of unlimited increase beyond the classical limits (unless at the expense of continuity at the classical libration limits or of finiteness at some singular point) and these values are the "characteristic values" to be used in the Bohr frequency condition. The sharpness of dropping off beyond the classical limits is greater the larger the quantum number or the smaller h (which for mathematical purposes need not be restricted to 6.55×10^{-27}), and becomes infinitely abrupt in the limit $h = 0$ which should correspond

[15] We take the classical normalizing integral over a half rather than complete period τ in order to have the particle pass through dx only once. We could, of course, take the normalizing integral over a complete period, but then the value of C would be halved and twice the expression (3) would give the probability of being in dx, as the particle might be there either in moving from left to right or from right to left, supposing the system oscillatory. When the variable of integration is changed to x the classical integral is between libration limits whereas the quantum one is from $-\infty$ to $+\infty$, but this distinction is immaterial because of the rapid exponential drop of the quantum probability beyond the classical libration limits if the quantum numbers are large.

to classical theory. The presence of a finite, though rapidly diminishing density beyond the classical libration limits when $h \neq 0$ gives rise to many interesting phenomena, as it shows that there is a probability, though small, of a particle passing beyond the classical libration limits and even surmounting a peak of potential energy greater than the total energy of the particle. This effect underlies the interesting quantum theory of nuclear disintegration given by Gurney and Condon and also simultaneously by Gamow,[16] and also the theory developed by Oppenheimer, Fowler and Nordheim [17] of the pulling of electrons out of cold metals by electric fields. This quantum probability of passage beyond the classical limits may be likened to the diffraction of light beyond the geometric shadow of a straight edge—wave mechanics and the undulatory theory of light respectively enable us to understand these two phenomena which would be totally incomprehensible with classical dynamics or geometrical optics.

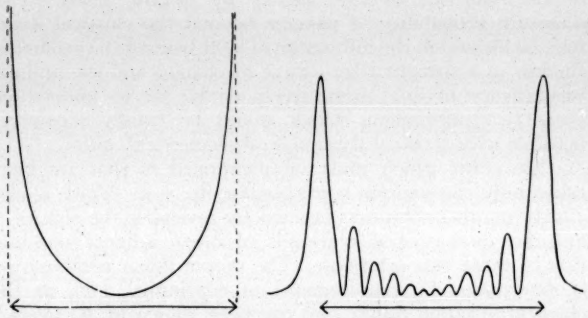
One point which must be mentioned is that we have taken only the solution corresponding to $+ i$. There is also clearly another solution obtained by reversing the sign of i, and the most general solution is, of course, a linear combination of these two solutions. The proper linear combination is determined by requirements of continuity, etc., at the classical libration limits, and has been shown by Kramers [18] to involve coefficients of equal absolute value for the solutions corresponding to $+ i$ and $- i$. This means that superposed on $Ae^{2\pi i S_1} = [A^2/2m(W - V)]^{1/4}$ we no longer have a factor $e^{2\pi i S_0/h}$ which can be disregarded between the classical limits because of its unit absolute value, and instead we have a periodic factor $\cos(2\pi S_0/h)$, whose square should be multiplied into the right side of (7). The quantum probability function then shows wave-like maxima and minima within the classical libration limits. It is the envelope of these waves, which is obtained by disregarding the cosine factor, which corresponds

[16] R. W. Gurney and E. U. Condon, *Nature*, Sept. 22, 1928; G. Gamow, *Zeits. f. Physik*, **51**, 204 (1928).

[17] J. R. Oppenheimer, *Proc. Nat. Acad. Sci.*, **14**, 363 (1928); Fowler and Nordheim, *Proc. Roy. Soc.*, **119**, 173 (1928); also Richardson, *ibid.*, **117**, 719 (1928).

[18] H. A. Kramers, *Zeits. f. Physik*, **39**, 829 (1928).

asymptotically with classical theory, but this modification need not worry us, as the successive maxima come closer and closer together as the quantum numbers become large or h small, making the irregularities due to the wave-like structure less and less important just as the distinction between physical and geometrical optics usually becomes less important with shorter wave-lengths. A graphical comparison of the classical and quantum distribution functions, in which the probabilities are taken as ordinates and the positions x as abscissa, would be somewhat as follows:



CLASSICAL. QUANTUM.

The classical libration limits are indicated by arrows. The classical probability becomes infinite, though integrable, at these limits as the particle moves infinitely slowly here. The quantum probability will not become infinite but will be large in this vicinity if the quantum numbers are large. The quantum curve is drawn for a state of quantum number 11, and is somewhat idealized as the agreement with classical theory might not be as perfect as indicated with so low a quantum number, since the terms $S_2, \cdots$ in the expansion (4) might have a perturbing effect. That the quantum number is 11 can be read directly from the figure because of the mathematical theorem [19] that the number of nodes of the wave function is the quantum number, i.e., index number of

[19] Courant-Hilbert, "Methoden der Mathematischen Physik," pp. 363–367.

the characteristic value with lowest value numbered zero. Without giving the rigorous proof, this is plausible in view of the fact that there was a factor $\cos [2\pi \int p dx/h]$ in our approximate expression for ψ.

This proof of the asymptotic identity given for a one-dimensional conservative system can be extended to systems which have any number of degrees of freedom, and which are non-conservative, so that W is replaced by the operator $h\partial \cdots /2\pi i \partial t$. The results are summarized in the following table, in which it is supposed for concreteness that the system is conservative and that the transformation is from a set of positional coördinates and momenta q, p to a set of action and angle variables J, w, making the Jordan-Dirac transformation theory reduce essentially to the Heisenberg and Schroedinger procedures.

CLASSICAL TRANSFORMATIONS.	QUANTUM ANALOG.
Given a contact transformation connecting two sets p, q and J, w of canonical variables. Then $p_k = \partial S/\partial q_k$, $w_k = \partial S/\partial J_k$ with	Let $\psi = \psi(q_1 \cdots q_f;\ J_1 \cdots J_k)$ be the wave function with $J_k = n_k h$ $(k = 1 \cdots f)$
$S = S(q;\ J) \longleftrightarrow (h/2\pi i) \log \psi + \text{const.}$	
If the energy W is a function only of the J's,	The wave equation for ψ is
$H(\partial S/\partial q;\ q) = W(J) \longleftrightarrow H(h\partial \cdots /2\pi i \partial q;\ q)\psi = W(J)\psi$	
(Asymptotic connection since $\frac{h}{2\pi i}\frac{\partial}{\partial q}[f(q)e^{2\pi i S/h}] = \frac{\partial S}{\partial q} f(q) e^{2\pi i S/h}[1 + h(\cdots) + \cdots]$)	
Assume the J's given and all values of the w's equally probable. Then the probability that the system be in any given configuration is proportional to the "lapse in phase"	Probability that the system be in the configuration $dq_1 \cdots dq_f$ is
$dw_1 \cdots dw_f = \Delta dq_1 \cdots dq_f \longleftrightarrow \lvert\psi\rvert^2 dq_1 \cdots dq_f$	
(Asymptotic connection since [20] $\psi = \Delta^{1/2} e^{2\pi i S/h}[1 + h(\cdots) + \cdots]$)	
Average value of any function $f(p(J, w);\ q(J, w))$ is	Corresponding average is a diagonal matrix element
$\overline{f(p, q)} = A \int \cdots \int f(p;\ q) dw_1 \cdots dw_f$ $= A \int \cdots \int f(\partial S/\partial q;\ q) \Delta dq \longleftrightarrow f(JJ) = \int \cdots \int \psi^* f(h\partial \cdots /2\pi i \partial q;\ q)\psi dq$	

In this table expressions which correspond asymptotically are connected by arrows, and Δ is used as an abbreviation

[20] J. H. Van Vleck, *Proc. Nat. Acad. Sci.*, **14**, 178 (1928).

for the functional determinant

$$\Delta = \frac{\partial(w_1, \cdots, w_f)}{\partial(q_1, \cdots, q_f)} = \left| \frac{\partial^2 S}{\partial q_i \partial J_k} \right|$$

also (p, q) and dq are abbreviations for $(p_1, \cdots, p_f; q_1, \cdots, q_f)$ and $dq_1 \cdots dq_f$, while S is similar to our previous S_0. For proof that the wave function does contain asymptotically the factor $\Delta^{1/2}$ as stated in the table the reader is referred to another paper in which I show that in the general case the first term in the series development (4) is the classical action function while $e^{2\pi i S_1}$ is proportional to $\Delta^{1/2}$. (In the simple one-dimensional example previously considered Δ was proportional to the radical $(W - V)^{-1/2}$.) It is to be particularly noted that it is necessary to include the second approximation S_1 in order to obtain the factor involving the functional determinant and thus establish the asymptotic identity of probabilities. This second term, unlike the still higher terms $S_2 \cdots$, does not give an effect which disappears in the limit $h = 0$ as the h contained in hS_1 just cancels the h in the denominator of the exponent in (4).

The last line of the above table deals with averages over all configurations rather than with the probability of being in one particular configuration. Most of our previous discussion of the asymptotic identity, especially in the one-dimensional example, has been primarily relative to the probabilities rather than averages, but the asymptotic connection of the classical statistical averages with the diagonal matrix elements is shown in virtually the same fashion, the proof hinging on the fact that the operator $h\partial \cdots / 2\pi i \partial q$ applied to $f(q)e^{2\pi i S/h}$ has the same effect in the limit $h = 0$ as multiplication by $\partial S/\partial q$, the classical value of p. In order to bring out more clearly the role of coördinates in the transformation theory the indices of the matrix in the last line of the table are taken as the action variables J rather than the quantum numbers n from which they differ by a factor h.

Although the above table is ostensibly for a set J, w of action and angle variables it applies equally well with any set α, β of canonical coördinates in place of J, w except that ψ becomes a more general "probability amplitude" or "transformation function" in place of the Schroedinger one,

and the differential equations for determining S and ψ are no longer necessarily the Hamilton-Jacobi and wave equations connected with the energy. It can be shown that the passage from the instantaneous values of variables specifying a dynamical system at a given time to their values at subsequent times can be regarded as a contact transformation. We have said that contact transformations have an analog in quantum mechanics but are associated with distribution or probability functions rather than with precise correlations. This means that the future of a particle is only statistically determined. The reason for the ambiguity is not necessarily to be imputed to breakdown of the "causality principle" but rather to the fact that the initial conditions are inevitably indeterminate because of Heisenberg's principle. This is indicated in the following diagram:

CLASSICAL DETERMINISM.	QUANTUM INDETERMINISM.
Given initially $x, y, z; v_x, v_y, v_z$.	Given v_x, v_y, v_z accurately; then x, y, z are undetermined (i.e., all values are equally probable for x, y, z); or vice versa x, y, z given and v_x, v_y, v_z undetermined.
↓ (Arrow indicates calculation by classical dynamics)	↓ (Arrow indicates calculation by quantum dynamics)
Accurate determination of x, y, z, v_x, v_y, v_z at all subsequent times.	Only statistical determination of x, y, z, v_x, v_y, v_z at subsequent times (i.e., probability of different values predicted).

These considerations may be illustrated by the scattering of alpha particles. Classically an alpha particle of given velocity is deflected through a certain angle whose magnitude depends on the closest distance of approach to the scattering center. On the other hand in quantum mechanics if the alpha particles have a definite vector velocity their positions are indeterminate by Heisenberg's principle, so that we cannot say how close a given alpha particle comes to the scattering center. However, the quantum mechanics does furnish directly formulas for the "distribution in angle" of the scattered alpha particles, which is what is observed experimentally; whereas the classical theory furnishes such formulas only with the intermediary of the non-observable distance of closest approach and with the supplementary statistical as-

sumption that all distances of approach are equally probable. Both the classical and quantum formulas for this distribution are virtually the same, luckily for the classical theory, which means that it is a tolerably good approximation to stop with the second term S_1 in the series (4).[21] On the other hand the inadequacy of classical theory in the Davisson Germer effect is due to the fact that free electrons, unlike alpha particles, have such long wave-lengths h/mv that quantum modifications and diffraction effects become important.

Difficulties of Existing Theory.—In closing I shall mention briefly what seem to be some of the outstanding difficulties and problems of quantum mechanics.

1. The first of these, probably not the most fundamental, is the choice of coördinates in the wave equation or Hamiltonian function. In the case of the hydrogen atom, for instance, one obtains the correct equation by substituting $h\partial \cdots/2\pi i\partial q$ for each momentum p in the classical Hamiltonian if one uses Cartesian coördinates but not if one uses polar.[22] Instead if one desires to use polar coördinates one must first set up the differential equation in Cartesian and then transform to polar. We have as yet no adequate explanation of why the extrapolation from classical coördinates to operators should be successful with Cartesian coordinates, or, more broadly, no adequate theory of the choice of Hamiltonian functions.

2. A second difficulty is that of the structure of the nucleus.[23] Certain nuclei, for instance, nitrogen, are known from atomic weights to contain probably an odd number of electrons, which according to ordinary spectroscopic theory

[21] Cf. J. R. Oppenheimer, *Phys. Rev.*, **31**, 72 (1928).

[22] If one uses polar coördinates, p_r^2 must be replaced by

$$(h/2\pi i)^2(d^2\cdots/dr^2 + 2r^{-1}d\cdots/dr)$$

with the ordinary choice of weight function in the definition of orthogonality. In an interesting paper (*Phys. Rev.*, **32**, 812, 1928) Podolsky points out that the reason a correct extrapolation to quantum mechanics cannot be obtained by a promiscuous substitution of momenta by operators is the ambiguity due to the non-commutativeness of quantum multiplications, but this, of course, does not explain the outstanding role of Cartesian coördinates.

[23] R. de L. Kronig, *Naturwissenschaften*, **16**, 335 (1928); Ornstein and van Wijk, *Zeits f. Physik*, **49**, 315 (1928).

would imply a paramagnetic behavior, whereas actually the nitrogen molecule is diamagnetic.[24] A somewhat related difficulty also pointed out by Kronig [23] is that Ornstein and van Wijk's measurements on alternating intensities in the N_2 band spectrum give a different intensity ratio between succeeding lines than would be predicted by the ordinary theory of statistical weights and number of particles in the nitrogen nucleus.

3. A third difficulty is with the relativity corrections, which are inextricably tied up with the subject of electron spin. Dirac [25] has very brilliantly devised a system of simultaneous first order linear equations which are tantamount to including both electron spin and relativity and which yield the desired Sommerfeld fine-structure formula to all degrees of approximation. Dirac's theory, however, has four wave functions per electron, which appears to be twice too many. Pauli [26] was able to interpret very nicely two wave functions per electron as the two values of a single function one of whose arguments could assume only two discrete values, one corresponding to an axial component $+\frac{1}{2}(h/2\pi)$ of spin angular momentum, and the other to $-\frac{1}{2}(h/2\pi)$. There is apparently no room for four wave functions per electron unless one interprets two of these as corresponding to a negative mass or positive charge. If the four wave functions were separable into two pairs which did not "combine" with each other, the difficulty would not be so perplexing, but actually this is not the case, so that there is a finite probability of the mass or charge of the electron changing sign, a most disquieting state of affairs!

4. Another, and perhaps related difficulty is that of spontaneous radiation. The only theory of this is Dirac's, which treats the ether as an ensemble of oscillators in a rather formal fashion suggestive of Jean's method of handling black body radiations. Undoubtedly Dirac's [27] procedure

[24] In this connection it is significant to note that Breit finds that Dirac's theory predicts an abnormally low magnetic moment at velocities comparable with that of light, such as might exist inside the nucleus. (*Nature*, **122**, 649, 1928.)

[25] P. A. M. Dirac, *Proc. Roy. Soc.*, **117A**, 610; **118A**, 351 (1928).

[26] W. Pauli, Jr., *Zeits. f. Physik*, **43**, 601 (1927).

[27] P. A. M. Dirac, *Proc. Roy. Soc.*, **114A**, 243, 710 (1927).

gives the correct method of attacking the problem from one angle, but presumably it should also be possible to give a treatment more nearly paralleling the classical theory of radiation associated with retarded potentials. Inherently connected with this is the question of the meaning of electromagnetic mass in quantum mechanics, as in classical theory the force of inertia due to electromagnetic mass and the force on an electron due to its own radiation are inextricably connected, being two succeeding terms in the same series development for the interaction between different parts of the electron.[28] Until we are able to transcribe into quantum language the classical analysis which made the mass of the electron inversely proportional to its radius it can scarcely be said that we have really solved the radiation problem in quantum mechanics. It is not impossible that an answer to this and especially to the preceding question of the physical interpretation of the Dirac relativity theory of the electron will not be obtained until we have a theory that explains the time-honored puzzle of "why is an electron," that is why is $e = 4.774 \times 10^{-10}$ and particularly why does a proton have a mass about 2,000 times greater than that of the electron.

5. A final difficulty is the cause underlying Pauli's exclusion principle. Why does nature have such a peculiar preference for antisymmetric wave functions?

I hope that mention of these five difficulties has not inculcated too much pessimism. The transformation theory and its statistical interpretation of quantum mechanics are, in my opinion, very perfect in their sphere of application, and most of the difficulties relate to phenomena beyond their present scope. The degree of perfection of the present quantum mechanics may perhaps be likened to that of classical analytical mechanics before electromagnetic theory and relativity.

[28] For slightly fuller elucidation of this statement and references to Lorentz's calculation, see footnote 5 of my "Quantum Principles and Line Spectra."

USES AND DEFECTS OF THE WAVE EQUATION.

BY

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THE masterly survey of the various formulations of quantum mechanics which Professor Van Vleck has given us seems to suggest that, if physicists in general are to understand the theory, some simplification in its formulation will be needed. After working with both transformation and wave theory, I have become convinced personally that much can be said in favor of adopting the de Broglie-Schrödinger-Heisenberg wave-theory as the fundamental formulation and regarding the matrix and transformation theories as special mathematical methods to be learned and used only by specialists in the mathematical theory. Apparently all results that possess physical significance can be obtained, and in most cases about as easily, in terms of the wave-theory.

A formulation in which the wave-theory is made fundamental has an advantage of formal character in that a much narrower fundamental postulate is required as to the physical interpretation. Consider, for instance, a particle in a conservative force-field, for which the wave equation can be written in either of two forms applicable to any conservative system:

$$\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = H\left(-\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right)\psi, \tag{1}$$

or

$$\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} + H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right)\psi = 0,$$

the values of ψ in the two cases being complex-conjugates of each other. The fundamental postulate as to physical interpretation consists in the statement that $\psi\psi^* dq$ is the probability that an exact observation of the position of the particle (for instance with cosmic-ray microscopes) would reveal it as having values of the coördinates within dq. The

fundamental rôle thus assigned to position fits in well with the fact that "position at a certain time" (the "event" of Relativity) is nearly always the thing actually observed in practice.

To obtain the probability for any other variable we then merely analyse a method of measuring that variable. Examples of such a procedure are to be found in papers by Darwin (e.g. *Roy. Soc. Proc.*, 117, 258, 1927) and Kennard (*Phys. Rev.*, 31, 876, 1928); Born's treatment of collision problems can be regarded as another. Let us consider, for instance, the treatment from this point of view of momentum. Defining it as $p = mv$, we may turn to the astronomer's method of measuring the velocity v. At any initial time, t, suppose that the probability amplitude ψ for a particle is sensibly zero outside of a certain region of diameter D; let us allow the particle to run until a time t' and then let us observe its position with precision. The distance traversed divided by the interval, $t' - t$, can then be taken as an observed value of v; the indefiniteness in the initial position introduces an uncertainty only of order $D/(t' - t)$, which can be reduced ad libitum by increasing t'. The result of such an observation is, of course, known only statistically at the instant t; the detailed analysis shows that we can take as the probability amplitude for the momentum of a free particle, as measured in this way, the usual expression of which the one-dimensional form is

$$M(p) = \frac{1}{\sqrt{h}} \int \psi(q) e^{(2\pi i/h)pq} dq.$$

If the particle is not free, difficulties arise. Perhaps we can, with Heisenberg and Bohr, imagine the forces to be suddenly removed for the purpose of an experimental observation; if this cannot be done, the correct conclusion would seem to be that momentum cannot in such cases be made the object of precise physical observation and so does not possess direct physical significance.

A similar analysis of an apparently plausible method of measuring the energy of a molecule leads to the result that the wave packet becomes spread out, in the space of the

center of mass of the molecule, into an "energy spectrum"; if the molecule is then brought to light experimentally at a certain point in this spectrum, its displacement from the zero position constitutes a mechanically observed value of the energy. The part of the original packet that comes to the point in the spectrum corresponding to the energy W_n of the nth quantum state contains the internal coördinates in the familiar harmonic standing-wave form, viz.:

$$\psi = \psi_1(x_c, y_c, z_c)e^{(2\pi i/h)W_n t}u_n(q),$$

$$H\left(-\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right)u_n(q) = W_n u_n(q),$$

x_c, y_c, z_c being the coördinates of the center of mass. We have thus a mechanical interpretation of W_n as a physically observable value of the energy.

The second point to which I wish to call attention is the peculiar nature of the imperfection which still exists in the wave-theory, just as it does in all other forms; for no form of quantum mechanics can yet deal quite satisfactorily with spontaneous transitions. Presumably the difficulty will disappear when the quantum theory of the electromagnetic field shall have been elaborated, and we shall then find that the completed wave-equation contains an additional term.

The new term must, however, be of a rather remarkable sort. Suppose, for instance, that we isolate an atom in a metastable state, in which it remains for a considerable interval until an instant t_j, when it jumps spontaneously into the normal state. By watching for the emitted photon we can actually determine experimentally the time at which the jump occurs. We know, therefore, that for quite a while ψ had the form, $\psi = \psi_1(q, t)$, ψ_1 being that solution of (1) which represents the metastable state; then, at time t_j, ψ suddenly ceases to obey this differential equation and changes quickly (perhaps instantaneously) into $\psi_2(q, t)$, the solution of (1) that is proper to the normal state. This statement about the change in ψ has physical meaning, for the values of $\psi_1\psi_1^*$ and $\psi_2\psi_2^*$ might be determined statistically by making many repetitions of the necessary observations; in each repetition we should make one determination of the

values of the q's, and in each case we should know whether we were testing ψ_1 or ψ_2 because we should know whether the photon had already been emitted or not. Obviously something is missing from the wave-equation as stated in (1); and of course the missing thing is simply the analogue of the radiation damping in classical theory. The forms so far proposed for this term produce, however, a steady change in ψ, beginning immediately; this is satisfactory for an ensemble of atoms, but does not cover the case of one atom as here described; nor is the latter case covered by the emission theory of Dirac. It would seem as if the missing term must be one whose values can be assigned only statistically; it must be a term which is usually zero but which has the power to assume temporarily very large values and so to cause a rapid change in ψ, at a time which is determined only by a probability law. It would seem that an element of probability must thus enter into the structure of the completed wave-equation itself.

STATISTICAL INTERPRETATION OF VARIOUS FORMULATIONS OF QUANTUM MECHANICS.

DISCUSSION BY

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AT the conclusion of his valuable paper Professor Van Vleck has concisely pointed out some of the outstanding difficulties in the present formulation of quantum mechanics.

To his first comment we would like to add that the main question does not seem to be so much the choice of the coördinates in the Hamiltonian as the "symmetrization" difficulty of Schrödinger. Cartesian coördinates enjoy a privileged position, just as in elementary geometry, because then the square of the distance between two points is a quadratic expression in diagonal form involving only the sum of squares. One can always find the wave equation in arbitrary coördinates by making use of the variational principle of Schrödinger. The ultimate reason for the symmetrization difficulty remains however unknown. (Cf. Schrödinger, *Annalen der Physik*, 79, p. 747, 1926.)

The problems raised by Dirac's theory of the electron spin are many and varied. It is well known that this theory starts out from the postulates, (*a*) that the wave equation must satisfy simultaneously the requirements of restricted relativity (invariance with respect to the Lorentz transformation) and (*b*) of the transformation theory (the wave equation should involve first order derivative with respect to time, i.e., the solutions of the wave equation form a normal orthogonal set). In addition Dirac requires (*c*) that for a vanishing field his iterated wave operator reduce to Schrödinger's relativistic wave equation. Thus the electronic spin in his theory appears as a consequence of relativity, or, alternately, as a consequence of combining relativity with the transformation theory; and it can be readily shown that the spin disappears in the limit for $c \to \infty$; more precisely the magnetic axis of the electron becomes unobservable.

Pauli has shown (*Zeits. für Phys.*, 43, p. 601, 1927) that, because of the fact that in a field only two orientations of the electron axis are possible, parallel and antiparallel to the field, two and no more than two functions are required to describe the spin. The fact that Dirac's equations involve twice as many functions is due to their undesirable characteristic that they apply equally well to an electron with a negative as with a positive charge—if there were such electrons. Further, owing to the fact that in general these four functions coexist, the probability for a change of sign of the electronic charge is finite and of the order of magnitude $(v/c)^4$ (v = velocity of the electron, c = velocity of light). In the case of the motion of an electron in a central field these four functions coalesce into two, which however refer just as well to the positive as to the negative electron. Under the circumstances it is rather unexpected to find that this theory is able to give Sommerfeld's fine structure formula exactly.

Darwin (*Proc. Royal Soc.*, 118, p. 658, 1928) has pointed out that, because of the lack of an adequate formulation of quantum electrodynamics, there is no quantum analogue to the classical calculation of electromagnetic mass. In fact the electronic mass appears in Dirac's theory as a consequence of his third requirement (c) (see above). Possibly this is connected with the fundamental question raised by Professor Van Vleck on the difference between the mass of the electron and that of the proton; certainly the absence of quantum electrodynamics has to do with the puzzles of spontaneous radiation. However, it can be readily shown that in the limiting case of vanishing electronic mass, and taking one of Dirac's functions constant, his equations go over, by applying a suitable canonical transformation, into Maxwell's equations in free space, just the same as Schrödinger's relativistic equation goes over into the light equation. In the same limit, and applying the same unitary (canonical) transformation, it can be shown that Dirac's electric density and electric current become the energy and the Poynting vector of Maxwell's theory. The statistical interpretation of these quantities, in agreement with that given to the corresponding ones in Dirac's theory, is then the probability that a light

quantum be in a certain region at a given time, and the probability that it cross a given boundary in a given direction, respectively. Further the change in the probability of its being within a region is equal to the probability of its having crossed the boundary of the region. It will be noted that while the energy and the Poynting vector satisfy the same continuity equation as the electric density and current, the former transform under the Lorentz group, not like a four-vector, but like the space-time components of the symmetrical energy-impulse tensor. While the algebra of the matrix operators involved in Maxwell's equations is that of quaternions, the Dirac operators involve an algebra connected with sedenions. (Cf. Felix Klein's "Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert," Vol. II, pp. 82, 85.)

The present transformation theory of quantum mechanics is based on the existence of a normal orthogonal set of (complex) functions and its related unitary transformations studied by Professor Weyl, and is also closely connected with Professor Wiener's coherency-matrix theory. Some of the conclusions arrived at by Professor Van Vleck in his paper find their adequate mathematical expression in the theory of coherency matrices, for example, the lack of *direct* physical reality of the non-diagonal elements of the matrices. There are great difficulties in formulating the transformation theory, which after all is molded on classical theory, in general relativity, for the transformations dealt with in the latter are not unitary transformations. Consequently the theory of orthogonal functions loses its privileged position. A generalization of Einstein's equivalence principle, or, mathematically, a generalization to function space of the theory of parallel displacement might remove this difficulty. A suggestion in this direction is contained in Einstein's paper on parallelism (*Berliner Sitzungsber.*, 1928, p. 217) where he introduces a system of four orthogonal axes h_a^ν at every point of a general four-dimensional manifold X_4 and obtains the gravitational potentials $g^{\nu\mu}$ ($\nu, \mu = 1, 2, 3, 4$) by summing on a ($a = 1, 2, 3, 4$). If to each point of the X_4 corresponds, not the tangential four-dimensional neighborhood, but a function space, we can obtain the $g^{\nu\mu}$ in an analogous way by taking four

complex functions $h^{\nu}(a)$ of a single variable in this space and integrating

$$g_{\nu\mu} = \int h_{\nu}(a)h_{\mu}(a)da.$$

This idea, which came to light during a conversation of the present authors with Professor Wiener, can also be considered as a generalization of certain considerations of Cartan ("Sur les variétés à connection projective," *Bull. de la Soc. Math. de France*, 52, p. 205, 1924). The $g_{\nu\mu}$ in this case are, however, positive, definite, and only a modification of the law of their formation can have a physical meaning. The application of Wiener's coherency matrices to the present case can be carried through without difficulty.

One has to distinguish between two kinds of statistical mechanics: that of an assemblage of point systems, in Gibbs' sense, and that of waves. Professor Van Vleck has pointed out one way of going over from one theory to the other. In either case one has to do with the study of certain averages, that is, with an integration problem over a certain range. One of the main problems here is the discovery of integral invariants, for example the $|\psi|^2$ of wave mechanics, or the number of particles in a given volume in classical statistical mechanics. Both are positive integral invariants.

THE PENDULUM PROBLEM IN THE WAVE MECHANICS.

BY

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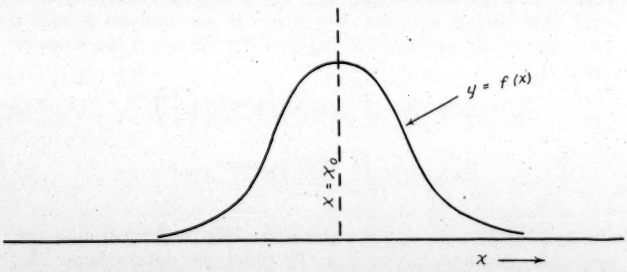
MY remarks have to do with an application and extension of the discussion of the one-dimensional oscillator problem given by Professor Van Vleck. They contain nothing particularly new but take up a simple illustrative problem which may be of interest.

Let us assume that we have a vibrating particle such as a pendulum with a known mass and known potential energy $V(x)$. We wish to predict the future position of the particle from an initial observation of position and corresponding

FIG. 1.



velocity. Assuming that the dimensions of the "particle" are large compared with the wave-length of light we may obtain the desired simultaneous initial observations from a photograph on a moving plate which will yield a graph of position as a function of time in the neighborhood of the time $t = 0$ at which the particle crossed the fiducial mark $x = x_0$. The slope of the trace at $x = x_0$ will give the initial speed and momentum of the particle. From the sharpness of the photograph, the uncertainty in the speed of the plate and other relevant information we may estimate

the probable errors x_1 and p_1 in the measured values of the initial position and momentum. We may even draw an error curve $y = f(x)$ with a maximum at $x = x_0$ to represent the probabilities of various actual initial positions or to represent the statistical distribution of initial positions in the case of a large number of experiments. With the drawing of a similar error curve for the momentum our information is complete.

We wish to determine the error curve for position at some future time and we do so with the aid of a function $\psi(x, t)$ called a probability amplitude. ψ is a solution of the Schrödinger differential equation and if properly adjusted to the initial conditions it has the property that $\psi\bar{\psi}dx$ is the probability that the particle has the coördinate x at the time t. ($\bar{\psi}$ is the complex conjugate to ψ.) Clearly $\psi(x, t_0)\bar{\psi}(x, t_0)$ must be identified with $f(x)$ to satisfy initial conditions.

But this one condition is insufficient to fix the form of the complex function $\psi(x, t_0)$. The momentum of the particle is linked with the wave-length λ of the ψ function in accordance with the famous formula $\lambda = h/p$. If we analyse ψ into a spectrum of monochromatic waves with the aid of the Fourier integral

$$\psi(x, t) = \int_{-\infty}^{+\infty} \varphi(\lambda, t) e^{2\pi i x/\lambda} d\left(\frac{1}{\lambda}\right) \tag{1}$$

$$\varphi(\lambda, t) = \int_{-\infty}^{+\infty} \psi(x, t) e^{-2\pi i x/\lambda} dx \tag{2}$$

the amplitude $\varphi(\lambda, t)$ plays the same rôle in determining the probable momentum as $\psi(x, t)$ plays in determining the probable position. If the momentum of the particle is sharply defined $\psi(x, t)$ will be approximately monochromatic and $\varphi(\lambda, t)$ will vanish except in a very narrow range of wave-lengths.

No general technique has been developed for fitting $\psi(x, t)$ to the initial experimental data for the momentum and hence we content ourselves with discussing the ideal case where the accuracy of the determination of the momentum is the maximum consistent with the wave picture or with the Heisenberg principle of uncertainty. We then make $\psi(x, 0)$

as nearly monochromatic as possible by giving it the value

$$\psi(x, 0) = \sqrt{f(x)}e^{-2\pi ix/\lambda_0} = \sqrt{f(x)}e^{-2\pi ip_0x/h}. \tag{3}$$

In order to follow the variation of $\psi(x, t)$ in time it is convenient to resolve it into a series of periodic solutions or characteristic functions.

$$\psi(x, t) = \sum_n c_n u_n(x) e^{2\pi i(W_n t/h + \epsilon_n)}. \tag{4}$$

Here the u's are the wave-like functions of x drawn by Professor Van Vleck and representable by the Kramers approximation formula

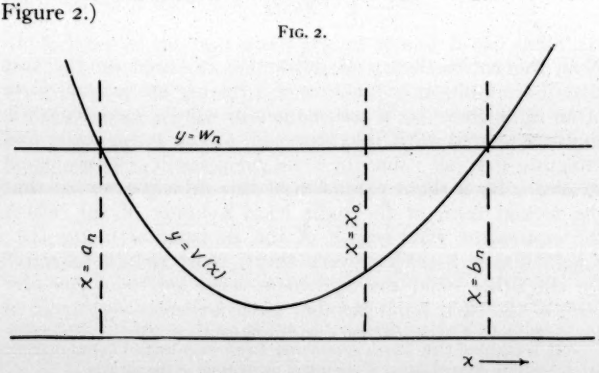
$$u_n(x) = A_n \sqrt{\lambda_n} \cos 2\pi \left[\int_{a_n}^{x} \frac{d\xi}{\lambda_n(\xi)} - \frac{1}{8} \right], \tag{5}$$

where

$$\lambda_n(x) = \frac{h}{|p_n(x)|} = \frac{h}{\sqrt{2m[W_n - V(x)]}}.$$

In (5) the action function S_0 of Professor Van Vleck's notation is replaced by its equivalent $\int_{a_n}^{x} h/\lambda_n d\xi$, where a_n is the lower of the two classical libration limits for the energy W_n. (Cf. Figure 2.)

FIG. 2.



Introducing the approximation function (5) into (4) and replacing the cosine by the sum of two exponentials we resolve ψ into two parts each representing a set of complex progressive waves, one moving to the right, the other to the left. Thus

$$\psi(x, t) = R(x, t) + L(x, t), \tag{6}$$

where

$$\left.\begin{aligned} R(x, t) &= \sum_n \frac{c_n A_n}{2} \sqrt{\lambda_n} e^{2\pi i \left\{ \frac{W_n}{h} t - \int_{a_n}^{x} \frac{d\xi}{\lambda_n} + \frac{1}{8} + \epsilon_n \right\}}, \\ L(x, t) &= \sum_n \frac{c_n A_n}{2} \sqrt{\lambda_n} e^{2\pi i \left\{ \frac{W_n}{h} t + \int_{a_n}^{x} \frac{d\xi}{\lambda_n} - \frac{1}{8} + \epsilon_n \right\}}. \end{aligned}\right\} \tag{7}$$

The constants c_n, ϵ_n are to be determined with the aid of the initial condition (3). By the usual rule for the expansion of a given function in terms of a series of orthogonal functions we have

$$\begin{aligned} c_n &= \int_{-\infty}^{+\infty} \psi(x, 0) u_n(x) e^{i\epsilon_n} dx \\ &= \frac{A_n}{2} \int_{-\infty}^{+\infty} \sqrt{\lambda_n} f e^{2\pi i \left[\frac{x}{\lambda_0} - \int_a^x \frac{d\xi}{\lambda_n} + \frac{1}{8} + \epsilon_n \right]} dx \\ &\quad + \frac{A_n}{2} \int_{-\infty}^{+\infty} \sqrt{\lambda_n} f e^{2\pi i \left[\frac{x}{\lambda_0} + \int_a^x \frac{d\xi}{\lambda_n} - \frac{1}{8} + \epsilon_n \right]} dx. \end{aligned} \tag{8}$$

Now the entire theory of diffraction is based on the fact that if we multiply a function of x having the general form of an error curve by a cosine function whose wave-length is short compared with the "spread" of the error curve and integrate over all values of x we get a result sensibly equal to zero. By a slight extension of this principle we see that the second term of the right hand member of (8), where the equivalent wave-length of the exponential factor (i.e., $\lambda_0\lambda_n(x)/[\lambda_0 + \lambda_n(x)]$) is very short, must reduce to zero.* On the other hand the equivalent wave-length of the first term of the right hand member of (8) becomes very large in

* It is assumed that $f(x)$ is spread over many wave-lengths (λ_0) as required for an accurate determination of the initial momentum of the particle.

the neighborhood of the initial position of the particle x_0 if $\lambda_n(x_0)$ is close to λ_0. Hence we may expect c_n to differ appreciably from zero only for a narrow range of n values which approximately satisfy the equation

$$\lambda_n(x_0) \equiv \frac{h}{\sqrt{2m[W_n - V(x_0)]}} = \frac{h}{|p_0|} \equiv \lambda_0. \tag{9}$$

The phase angles ϵ_n are determined to make the c_n's real which is accomplished by making the phases of all the constituents of $R(x, 0)$ agree with the phase factor in equation (3) at the starting point $x = x_0$. Analytically

$$\frac{x_0}{\lambda_0} = \int_{a_n}^{x_0} \frac{d\xi}{\lambda_n(\xi)} + \frac{1}{8} + \epsilon_n.$$

With the aid of this explicit formula for ϵ_n the expressions for the forward and backward traveling wave packets become

$$R(x, t) = \sum_n \frac{c_n A_n}{2} \sqrt{\lambda_n} e^{2\pi i \left\{ \frac{W_n}{h} t - \int_{x_0}^{x} \frac{d\xi}{\lambda_n} - \frac{x_0}{\lambda_0} \right\}}, \tag{10}$$

$$L(x, t) = \sum_n \frac{c_n A_n}{2} \sqrt{\lambda_n} e^{2\pi i \left\{ \frac{W_n}{h} t + \int_{a}^{x} \frac{d\xi}{\lambda} + \int_{a}^{x_0} \frac{d\xi}{\lambda} - \frac{1}{4} - \frac{x_0}{\lambda_0} \right\}}. \tag{11}$$

Neither of the two wave groups R and L can differ appreciably from zero except for values of x and t which make the phases of the constituent waves approximately the same. At the initial instant this condition is satisfied by R in the neighborhood of x_0 but by L nowhere. For R the condition is expressed by the equation

$$\frac{d}{dn}\left[\frac{W_n}{h} t - \int_{x_0}^{x} \frac{d\xi}{\lambda_n(\xi)} \right] = 0, \tag{12}$$

where n is to have the value given by (9) and corresponding to the most probable energy of the particle. The differential equation for the values of t and x which satisfy (12) is

$$\frac{dx}{dt} = \frac{dW_n}{dn} \div h \frac{d}{dn}\left(\frac{1}{\lambda_n} \right) = \frac{dW_n}{dp_n}. \tag{13}$$

As this is one of the classical Hamiltonian equations for a particle with energy W_n it is evident that the initial wave packet will move to the right with the classical speed until it reaches the libration limit b. At that point the wave-length becomes infinite and the approximation function breaks down. Moreover the constituent waves of R get hopelessly out of phase as t continues to increase and so R fades out of the picture.

At this point, however, the phases of the backward moving waves come into harmony at b and a backward moving packet travels toward a with the classical speed. It is perhaps sufficient to show that at $x = b$ each forward travelling wave agrees in phase with the corresponding backward wave. In other words

$$\int_a^b \frac{d\xi}{\lambda_n} + \int_a^{x_0} \frac{d\xi}{\lambda_n} - \frac{1}{4} - \frac{x_0}{\lambda_0} + \int_{x_0}^b \frac{d\xi}{\lambda_n} + \frac{x_0}{\lambda_0} = 0.$$

This statement is equivalent to the "Sommerfeld" quantum condition that

$$\oint p_n d\xi = 2h \int_a^b \frac{d\xi}{\lambda_n} = n + \frac{1}{2}$$

and is automatically fulfilled for each of the characteristic energy values W_n.

In closing, I may remark that the foregoing discussion of the one-dimensional wave packet problem lacks the elegance of the virial method of Ehrenfest and Ruark. It has, however, a slight advantage in that it shows that the packet will remain concentrated for a considerable length of time and it makes clear the close connection between this mechanical problem and the elementary theory of optical diffraction.

THE PROBLEM OF SYMMETRY IN QUANTUM MECHANICS.

BY

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AN atom or an ion, whose nucleus is considered as a fixed center of force O, possesses two kinds of symmetry properties: (1) the laws governing it are spherically symmetric, i.e., invariant under an arbitrary rotation about O, (2) it is invariant under permutation of its ν electrons. The first type of symmetry is described by the continuous group γ of orthogonal transformations of the spacial coördinates xyz and the second by the finite symmetric group of order $\nu!$ which consists of all permutations of ν things. Quantum mechanics represents the state of a given physical system by means of the vectors—or rather the rays—of an associated unitary system space $\mathfrak{R}_n$. The dimensionality n of $\mathfrak{R}_n$ is in general infinite, unless we restrict ourselves to states in which the energy, for example, possesses a definite value. The transformations which allow us to pass from one normal coördinate system to another such in unitary vector space are the unitary transformations, under which the fundamental form

$$\bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n$$

is invariant. I can now indicate how the representations of the symmetry group are introduced into quantum theory; I take the first type of symmetry as an instance. On subjecting the atom to the virtual rotation s an arbitrary state $\mathfrak{x}$ of the atom goes over into a new state $\mathfrak{x}'$; all relations and properties which have an objective significance must remain unaltered under s, and the transition $\mathfrak{x} \to \mathfrak{x}'$ must therefore constitute a unitary transformation $U(s)$ of system space into itself. It is furthermore clear that on compounding two rotations s the corresponding transformations $U(s)$ are likewise compounded

$$U(st) = U(s)U(t).$$

And this is exactly what we mean when we say that the correspondence $s \to U(s)$ is a unitary representation of the group γ in system space.

From the physical standpoint there exists an essential difference between the two kinds of symmetry which we have mentioned; the spherical symmetry can be destroyed by external influences, but the intrinsic identity of the electrons can not. If, for example, the atom is in a homogeneous magnetic field the rotational symmetry of the laws to which it is subject is reduced to the symmetry described by the group of rotations about the direction of the magnetic field. We shall in the following be exclusively concerned with that indestructible symmetry which is due to the intrinsic identity of all electrons.

I shall first develop the fundamental concepts of the theory of groups which we need for our present purpose. The vector spaces in which we operate are always unitary; only normal coördinate systems are allowed and all linear transformations are unitary, even though we do not mention it explicitly. The two fundamental operations on vector spaces are *additive* and *multiplicative composition*. If

$$\mathfrak{x} = (x_1, \cdots, x_m) \qquad \mathfrak{y} = (y_1, \cdots, y_n)$$

be vectors in the m- and n-dimensional vector spaces $\mathfrak{R}$, $\mathfrak{S}$ respectively, we consider the pair $(\mathfrak{x}, \mathfrak{y})$ with $m + n$ components $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ as a vector in a new $(m + n)$-dimensional space $\mathfrak{R} + \mathfrak{S}$. A linear transformation $\mathfrak{x} \to \mathfrak{x}' = A\mathfrak{x}$ in $\mathfrak{R}$ and a linear transformation $\mathfrak{y}' = B\mathfrak{y}$ in $\mathfrak{S}$ together constitute a linear transformation A,B in the total space, which is conversely decomposed into the transformations A and B of the linear sub-spaces $\mathfrak{R}$, $\mathfrak{S}$ of $\mathfrak{R} + \mathfrak{S}$. It is in fact, this process of addition which allows us to generate an n-dimensional vector space from the one-dimensional, which is nothing other than the continuum of complex numbers.

On the other hand, we consider the totality of products $x_i y_k = z_l$ as the components of a vector $\mathfrak{x} \times \mathfrak{y}$ in an (mn)-dimensional space $\mathfrak{R} \times \mathfrak{S}$. l is an abbreviation for the pair (i, k) which serves to indicate the various vector components in the product space. Under the influence of linear transformations A and B of x_i and y_k the products z_l undergo a

linear transformation which we denote by $A \times B$. This process is the foundation of tensor analysis; a tensor of third order in the vector space $\mathfrak{R}$ for example, is the same as a vector in $\mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} = \mathfrak{R}^3$ and under the influence of a linear transformation A in $\mathfrak{R}$ the tensor components undergo the corresponding transformation $A \times A \times A$ of $\mathfrak{R}^3$. Multiplicative composition is of utmost importance in quantum theory for the following reason: If $\mathfrak{R}$ be the system space of a physical system P and $\mathfrak{S}$ that of the system Σ, then $\mathfrak{R} \times \mathfrak{S}$ is the system space of the total system composed of the two parts P, Σ. The special case considered above—that of repeated $\times$-multiplication of $\mathfrak{R}$ with itself—is of particular interest for our present discussion: if $\mathfrak{R}$ be the system space of a single electron moving in the field of the nucleus the state of each of the ν electrons of the atom can be represented as a vector in the same system space $\mathfrak{R}$ because of the likeness of the electrons, and the state of the atom is therefore represented by a vector in $\mathfrak{R} \times \mathfrak{R} \times \cdots \times \mathfrak{R}$ (ν factors) or by a tensor of order ν in $\mathfrak{R}$.

But to return to the theory of groups, let γ be a given group and let s run through the elements of γ. If $\mathfrak{a} : s \to A(s)$ be a unitary representation of γ of degree m (i.e. a unitary representation in the m-dimensional vector space $\mathfrak{R}$) and $\mathfrak{b} : s \to B(s)$ a representation in the n-dimensional vector space $\mathfrak{S}$ then $s \to A(s), B(s)$ is a representation in the $(m + n)$-dimensional space $\mathfrak{R} + \mathfrak{S}$. We designate this representation by $\mathfrak{a} + \mathfrak{b}$. Conversely $\mathfrak{a} + \mathfrak{b}$ is decomposed into the two independent constituents $\mathfrak{a}$ and $\mathfrak{b}$. A representation is said to be *primitive* if it cannot be so decomposed by any choice whatever of the normal coördinate system in system space. Two representations which differ only in the coördinates in terms of which they are expressed are said to be *equivalent*. The *first fundamental problem* for the theory of representations is *to construct a complete system of inequivalent primitive representations of the given group* γ. The solution of this problem for the symmetric group is due for the most part to G. Frobenius. Again, the product $\mathfrak{a} \times \mathfrak{b}$ of two representations $\mathfrak{a}$ and $\mathfrak{b}$ is defined by the correspondence $s \to A(s) \times B(s)$. The *second fundamental problem* is: *Given the primitive representations* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* γ, *to decompose* $\mathfrak{a} \times \mathfrak{b}$ *into its primitive constituents.*

The trace of $A(s)$—the sum of the elements in the principal diagonal of the matrix $A(s)$—is the group-characteristic, or simply the character $\chi(s)$ of the representation $\mathfrak{a}$; equivalent representations have the same characters. The converse of this theorem is valid for unitary representations, and we can consequently employ the character to designate the representation. The character of $\mathfrak{a} + \mathfrak{b}$ is the sum of the characters of $\mathfrak{a}$ and $\mathfrak{b}$, and that of $\mathfrak{a} \times \mathfrak{b}$ is their product. We always have

$$\chi(s^{-1}) = \bar{\chi}(s).$$

The characters of inequivalent primitive representations constitute an orthogonal system

$$\mathfrak{M}\{\bar{\chi}(s)\chi(s)\} = 1, \qquad \mathfrak{M}\{\bar{\chi}(s)\chi'(s)\} = 0$$

where χ, χ' are inequivalent and $\mathfrak{M}$ denotes the mean value. In this formulation I have in mind primarily finite groups—such as the symmetric group—but the theory is also valid for closed continuous groups; the mean value is then obtained by integrating over the group manifold instead of by summation. If the representation $\mathbf{X}$ contains the primitive representation χ exactly g times, the primitive representation χ' g' times, etc., then

$$\mathbf{X}(s) = g\chi(s) + g'\chi'(s) + \cdots.$$

The coefficients g are uniquely determined by this equation; e.g. we have

$$(*) \qquad g = \mathfrak{M}\{\mathbf{X}(s)\bar{\chi}(s)\}.$$

The decomposition of a representation $\mathbf{X}$ into its primitive constituents is consequently unique (in the sense of equivalence) and equation (*) shows how many times the primitive representation χ occurs in $\mathbf{X}$. In particular this equation gives the solution of our second fundamental problem: The number of times that the primitive representation χ occurs in the cross product of the two primitive representations χ_1, χ_2 is given by the mean value

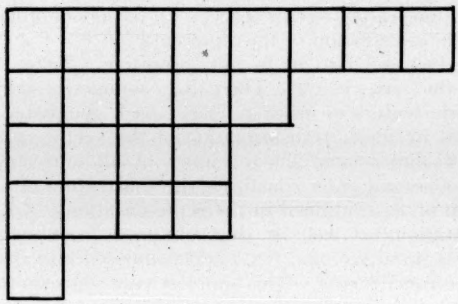
$$g = \mathfrak{M}\{\chi_1(s)\chi_2(s)\bar{\chi}(s)\}.$$

We take as an example the group Γ_n of all unitary trans-

formations in n dimensions, a case of great importance for physics. Since Γ_n itself consists of linear unitary transformations, the correspondence $s \to s$ constitutes a representation — in fact, a primitive representation — $\mathfrak{i}$, the field of operation or *substratum* of which is the aggregate of *vectors* of the n-dimensional vector space. All tensors of order ν constitute the substratum of the representation $\mathfrak{i} \times \mathfrak{i} \times \cdots \times \mathfrak{i} = \mathfrak{i}^\nu$ (ν factors) but it is not primitive. Take, for example, the case $\nu = 2$. There are symmetric and anti-symmetric tensors of order 2. Since such symmetry is not destroyed by linear transformation of the vector space, the $\frac{1}{2}n(n-1)$-dimensional linear family of all anti-symmetric tensors of second order constitute the substratum of a representation of Γ_n contained in the representation $\mathfrak{i} \times \mathfrak{i}$. This representation, as well as that obtained from symmetric tensors, is primitive, and $\mathfrak{i} \times \mathfrak{i}$ is decomposed into these two primitive constituents. This simplest case suggests that the substrate of the primitive representations contained in $\mathfrak{i}^\nu$ are *tensors of order ν with certain definite symmetry properties.* We must therefore examine the various types of symmetry possible in tensors of order ν. But this problem is closely related to that of the representations of the group σ_ν of permutations of ν things. This can be shown quite readily, but I shall content myself with giving the result. With each representation $\mathfrak{d}$ of σ_ν there is associated a definite representation $\mathfrak{G}$ of Γ_n of order ν. This correspondence is to be understood in the sense that on both sides equivalent representations are to be taken as indistinguishable. $\mathfrak{G}$ is primitive if $\mathfrak{d}$ is; if $\mathfrak{d}$ be of degree f then $\mathfrak{G}$ occurs exactly f times in $\mathfrak{i}^\nu$. If χ be the group-characteristic of $\mathfrak{d}$, I designate $\mathfrak{G}$ more precisely by $\mathfrak{G}_\chi$. $\mathfrak{R} \times \mathfrak{R} \times \cdots \times \mathfrak{R}$ is decomposed into a number of primitive sub-spaces $\mathfrak{R}_\chi^{\nu}$ which are independent of the coördinate system employed, and the group of all unitary transformations in $\mathfrak{R}$ induces in each of these a primitive representation $\mathfrak{G}_\chi$. This correspondence between the representations $\mathfrak{d}$ and $\mathfrak{G}$ of σ and Γ must be expressible by a simple relation between their characters χ and X. I shall not write down this formula, but I shall have occasion to refer to it later and shall call it the "bridge"; that is, the bridge which connects the representations of σ and Γ.

Since continuous groups are much easier to handle than discrete, the "bridge" leads over X to an explicit determination of the primitive character χ of the permutation group σ_ν.

Diagram $\mathfrak{S}$



I have yet to indicate the simple construction which enables us to obtain the tensors of order ν of highest possible symmetry. By a permutation s a tensor F of order ν is changed into sF. On subjecting an arbitrary tensor F to all possible permutations s we can generate the most general symmetric tensor by the formula $\Sigma_s sF$ and the most general antisymmetric one by $\Sigma_s \pm sF$. The sign $\delta_s = \pm$ is $+$ if s be an even permutation and $-$ if it be odd. We call the first process symmetrization and the second alternation. We now write the numbers from 1 to ν in a diagram $\mathfrak{S}$ which consists of non-interrupted rows and columns as in the accompanying figure. An arbitrary tensor is subjected to symmetrization with respect to the figures of each of the rows and then to alternation with respect to the figures of each of the columns. The tensors so found constitute a linear family which is the substratum of a primitive representation $\mathfrak{G}_\chi$ of Γ of order ν, which is determined by the diagram $\mathfrak{S}$, and to this there corresponds a definite primitive representation $\mathfrak{d}$ of σ_ν. Different diagrams give rise to inequivalent representations. If the diagram consists of but one row, we obtain the family of all symmetric tensors; the corresponding

representation of σ_ν is one-dimensional and associates the identity $x' = x$ with each permutation s. If the diagram consists of but one column we obtain antisymmetric tensors; the corresponding representation of σ_ν is again one-dimensional and associates the transformation $x' = \delta_s x$ with the permutation $s : \chi(s) = \delta_s$. On interchanging rows and columns in $\mathfrak{S}$ we obtain the dual diagram $\mathfrak{S}^*$; the corresponding characteristics are related by

$$\chi^*(s^{-1}) = \delta_s \chi(s) \quad \text{or} \quad \bar{\chi}^*(s) = \delta_s \chi(s).$$

The primitive sub-spaces $\mathfrak{R}_\chi^\nu$ of $\mathfrak{R} \times \mathfrak{R} \times \cdots \times \mathfrak{R}$ have the following significance for quantum physics: If the state of an atom is represented by a vector in $\mathfrak{R}_\chi^\nu$ no influence can drive it out of $\mathfrak{R}_\chi^\nu$. To each sub-space there belongs a certain symmetry-class of spectral terms; equivalent sub-spaces—in which Γ_n induces equivalent primitive representations—yield the same terms. Corresponding to the primitive representation χ (of degree f) of the permutation group there is a single class of terms whose members are f-fold, and it is impossible to resolve this hidden multiplicity. The various primitive sub-spaces are, so to speak, worlds which are fully isolated from one another. But such a situation is repugnant to Nature, who wishes to relate everything with everything. She has accordingly avoided this distressing situation by annihilating all these possible worlds except one—or better, she has never allowed them to come into existence! The one which she has spared is that one which is represented by anti-symmetric tensors, and this is the content of Pauli's exclusion principle. The system space of an atom is accordingly not the space $\mathfrak{R}^\nu$ of all tensors of order ν in $\mathfrak{R}$ but only the sub-space of anti-symmetric tensors. Our entire theory thus appears to be quite superfluous for physics—were it not for the spinning electron!

Briefly and vaguely stated, this amounts to the following: the electron is in the abstract composed of two partial systems, the electron translation and the electron spin. Denoting the corresponding system spaces by $\mathfrak{R} = \mathfrak{R}_n$ and $\mathfrak{R}' = \mathfrak{R}_a$ the system space of the electron is $\underline{\mathfrak{R}} = \mathfrak{R}' \times \mathfrak{R}$. The dimensionality n is infinite, but the dimensionality a appears to be 2. But the dynamical laws governing the state vector in $\underline{\mathfrak{R}}$

involve to first approximation (on neglecting the small perturbation due to the spin) only the translational part $\mathfrak{R}$ of $\underline{\mathfrak{R}} = \mathfrak{R}' \times \mathfrak{R}$. Let $\Gamma_a = \Gamma'$, $\Gamma_n = \Gamma$ and $\underline{\Gamma}$ denote the groups of unitary transformations in $\mathfrak{R}_a$, $\mathfrak{R}_n$ and the product space $\mathfrak{R}_a \times \mathfrak{R}_n$ and u', u the elements of the first two. Let $\mathfrak{G}'_{\chi'}$ be a primitive representation of $\Gamma' : u' \to U'$; $\mathfrak{G}_\chi$ of $\Gamma : u \to U$. Then the correspondence

$$(u' \times u) \to (U' \times U)$$

yields a primitive representation of $\Gamma' \times \Gamma$ which we designate by $\mathfrak{G}'_{\chi'} \otimes \mathfrak{G}_\chi$; its character belonging to the element

$$u' \times u \quad \text{is} \quad \chi'(u')\chi(u).$$

The quantum-theoretic symmetry problem is now: $\mathfrak{G}_\xi$ being a primitive representation of $\underline{\Gamma}$, to determine how $\mathfrak{G}_\xi$ is decomposed into primitive constituents of the form $\mathfrak{G}'_{\chi'} \otimes \mathfrak{G}_\chi$ on restricting $\underline{\Gamma}$ to the sub-group $\Gamma' \times \Gamma$. We can add this third problem, formulated generally, to the above mental-funda problems. The "bridge" reduces it to the second fundamental problem for the symmetric group, by means of the following reciprocity: $\mathfrak{G}_\xi$ contains the representation $\mathfrak{G}'_{\chi'} \quad \mathfrak{G}_\chi$ on restricting $\underline{\Gamma}$ to $\Gamma' \times \Gamma$ as many times as the representation of $\mathfrak{d}_{\chi'} \times \mathfrak{d}_\chi$ contains the representation $\mathfrak{d}_\xi$. But in accordance with (*) this multiplicity is given by

$$g = \mathfrak{M}\{\chi'(s)\chi(s)\bar{\xi}(s)\}.$$

The quantum theory is only interested in the case where $\mathfrak{G}_\xi = \{\mathfrak{G}\}$ is the Pauli representation which is associated with the character $\zeta(s) = \delta_s$. But $\chi(s)\delta_s$ is by the above $\bar{\chi}^*(s)$ and in consequence of the orthogonal properties of $\chi(s)$ mentioned above $g = 1$ or 0 according as χ' is the dual of χ or not. The representation induced by the group $\Gamma' \times \Gamma$ in the Pauli system-space of the atom contains only primitive constituents of the form $\mathfrak{G}'_{\chi^*} \otimes \mathfrak{G}_\chi$ and these but once. Finally, we must replace the element u' (which has so far been arbitrary) of Γ' by the identity. On thus restricting ourselves to the transformations $1 \times u$ of $\underline{\Gamma}$, $\{\mathfrak{G}\}$ becomes a representation of Γ whose decomposition into irreducible constituents is given, in accordance with our development, by

$$\sum_\chi F_a(\chi^*)\mathfrak{G}_\chi,$$

where $F_a(\chi^*)$ means the degree of $\mathfrak{G}'_{\chi^*}$, i.e. the degree of the representation of the a-dimensional linear group which corresponds to the dual character χ^*. Again restricting ourselves to the electron translation, as is natural when the spin perturbation is neglected, the Pauli exclusion principle together with the existence of spin produces the following modifications of our original theory: The multiplicity of the term class χ is no longer $f(\chi)$, the degree of the irreducible representation χ of the permutation group, but is now $F_a(\chi^*)$. In particular, all term classes whose symmetry diagrams possess more than a columns do not occur. The various classes of terms exist side by side, for their isolation is destroyed by the spin-perturbation which gives rise to weak intercombinations; furthermore, the spin-perturbation resolves the "multiplets" of a class into a number of simple components which is given by the degree $F_a(\chi^*)$.

The reciprocity indicated above is concerned with the case in which Γ_{an} is restricted to $\Gamma_a \times \Gamma_n$, but we can set up the analogue for the permutation group. Divide the numbers from 1 to ν into two classes containing ν' and ν'' integers: $\nu' + \nu'' = \nu$. An arbitrary permutation p' of the numbers in the first class and an arbitrary permutation p'' of those in the other class together constitute a permutation of all the numbers which is designated by $p' \times p''$ in accordance with our convention; these special permutations constitute a sub-group $\sigma' \times \sigma''$ of $\sigma = \sigma_\nu$. If $\mathfrak{d}_\xi$ be a primitive representation of σ, then on restricting σ to the sub-group $\sigma' \times \sigma''$, $\mathfrak{d}_\xi$ is decomposed into certain primitive constituents of the form $\mathfrak{d}'_{\chi'} \otimes \mathfrak{d}''_{\chi''}$. The factors $\mathfrak{d}'$, $\mathfrak{d}''$ are representations of σ' and σ'' respectively. And the second reciprocity theorem to which we are immediately led by the "bridge," is: $\mathfrak{d}'_{\chi'} \otimes \mathfrak{d}''_{\chi''}$ appears in $\mathfrak{d}_\xi$, if considered as a representation of the subgroup $\sigma' \times \sigma''$, exactly as many times as the representation $\mathfrak{G}_{\chi'} \times \mathfrak{G}_{\chi''}$ of the unitary group Γ contains the primitive representation $\mathfrak{G}_\xi$ of the same.

This reciprocity law governs the fundamental chemical problem of combining two atoms to obtain a molecule. Let the number of electrons contained in the two atoms and the molecule be ν', ν'' and $\nu = \nu' + \nu''$, and let the atoms be in the symmetry states characterized by the permissible

representations χ', χ'' of σ', σ''. The molecule which is obtained by combining the two atoms will be in one of the symmetry states ζ whose corresponding $\mathfrak{G}_\zeta$ appears in $\mathfrak{G}_{\chi'} \times \mathfrak{G}_{\chi''}$ and the calculation of the associated energy is accomplished with the aid of the characteristics. These circumstances, which cannot be represented by a spacial picture, constitute the basis for the understanding of the homeopolar bond, the attraction (or repulsion) existing between neutral atoms. The second fundamental problem as applied to the unitary group Γ is the quantum-theoretic formulation of that combinatorial problem which the chemists describe by the aid of valence bonds.

Unfortunately it has not been possible to show here how completely the mathematical methods of the theory of representations lend themselves to these physical applications. The connection between mathematical theory and physical application which are revealed in the work of Wigner, v. Neumann, Heitler, London and the speaker is here closer and more complete than in almost any other field. The theory of groups is the appropriate language for the expression of the general qualitative laws which obtain in the atomic world. In particular the reciprocity laws between the representations of the symmetry group σ_f and the unitary group Γ are the most characteristic feature of the development which I have here indicated; they have not previously come into their own in the physical literature, in spite of the fact that quantum physics leads very naturally to this relation.

THE ROLE OF GROUPS IN A PHYSICAL THEORY

DISCUSSION BY

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PROFESSOR SLATER mentioned this morning that in theoretical physics we have a series of physical facts and a series of mathematical statements that run parallel to the facts. If we want to discuss the particular character of application of the Theory of Groups we have to begin by reformulating and amplifying this statement of Professor Slater.

A physicist, we may take it, is a person who measures according to certain rules. Let us denote by a the number he obtained in a given situation by applying the rule number one, by b the number obtained in the same situation by measuring according to rule number two and so on (a may be e.g. the volume, b the pressure, c the temperature of gas in a given container). The physicist finds further that the results of measurements of the same kind undertaken in different situations satisfy certain relations, we may write for instance

$$r(a, b) = c.$$

A mathematician is busy deducing from some given propositions other propositions; this usually leads to numbers which we may call A, B, $C \cdots$. These numbers also satisfy certain relations, say

$$R(A, B) = C.$$

Then comes, as Professor Weyl says, a messenger, a go-between who may be a mathematician or a physicist, or both, and says: "If you establish a correspondence between the physical quantities and the mathematical quantities, if you assign A to a, B to b, etc., the *same* relations hold for the physical quantities as for the corresponding mathematical quantities so that $R \equiv r$."

In the course of time new procedures of measurement are invented, some physical relations do not find their counterpart in the mathematical theory, the mathematical theory has to be patched up by introducing new quantities till too many quantities appear in it which do not correspond to physical quantities; then comes the phenomenological point of view and sweeps the theory out of applied mathematics—the theory becomes pure mathematics once more, and physicists begin to look around for a new theory. Everybody can find examples for this situation; it is enough to mention the Bohr atom which was not even mentioned today only 15 years after its introduction.

However the theory of groups which is being applied to physics is not just one of many mathematical theories of the character described above; its application is of a far more fundamental nature and we shall be able to indicate what it is by analysing further the scheme outlined above.

It may happen, and in fact it happens often, that the same mathematical theory can be applied to the same physical facts in more than one way; for instance, instead of assigning to the physical quantities $a, b, \cdots$ the mathematical quantities $A, B, \cdots$ we might have assigned to them $A', B', \cdots$ with the same results, that is, the relations for physical quantities are the same as for the mathematical quantities corresponding to them now (think of space considered from the experimental point of view—and of coördinate geometry; different ways of establishing a correspondence result from different choices of coördinate axes). If this happens it means that the mathematical theory possesses a peculiar property, namely, that if A' is substituted for A, B' for B and so on, no relation of the type $R(A, B) = C$ which was correct before the substitution is destroyed; in other words, there are substitutions or transformations for which all relations are invariant. All such transformations constitute what we call a group; the existence and the properties of such a group present a very important characteristic of the mathematical theory. Moreover it is clear that if two different mathematical theories can be applied—in the sense described above—to the same physical theory, the groups of these two theories will be essentially the same, so that the

groups reflect some of the most fundamental properties of physical systems.

In Quantum Mechanics it seems that some groups have been discovered and put to work with considerable success before the complete theories are ready. It is to be hoped, however, that some satisfactory complete theories will be offered soon; because, after all, the application of groups inspite of its fundamental character may be of restricted scope; not all essential features of a mathematical theory need be reflected in the group; even in geometry—the classical example of the application of groups—further refinement has lead to such changes (in the General Relativity Theory) that the group was reduced to nothing. There are indications that in Quantum Mechanics the groups cannot take care of everything: in one case Professor Weyl thought it necessary to invoke the wishes of Nature; it may be thought desirable to substitute for that another type of explanation, and we have to strive to arrive at it.

Preliminary Results of Ocean Magnetic Observations on the Carnegie. J. P. AULT. (*Ter. Mag.*, Sept., 1928.) On May 1, 1928 the *Carnegie* left her wharf in the city of Washington to begin a new series of scientific observations on the oceans of the earth. On the way to England the weather was unfavorable, it being necessary to wait ten days off the entrance to the English Channel on account of gales. The copper sheathing started to peel off at the water-line and both cabin and staterooms were flooded at times.

At 29 stations on the voyage observations of declination were made with the marine collimating compass. Chart corrections ranging from + 1° .7 to − 1° .0 were found for declination, while the largest error for inclination amounted to 0° .9. The vessel was stopped at six stations on the only days when the weather permitted and observations of the temperature of the water were made and samples of the water were collected. Water-bottles at one place failed to operate. "Some animal of the deep had fouled the wire."

From England the *Carnegie* went to Hamburg, where Dr. Kolhoerster delivered an instrument for observing the penetrating radiation associated with his name and that of Millikan in the United States. Thence the voyage led toward the Faroe Islands and on to Iceland. Off Iceland the vessel crossed her track of 1914. Reykjavik was at last reached on July 20.

Thus has begun the long voyage of the *Carnegie* that will furnish fresh data for magnetic charts of the water areas of the globe and throw light upon the secular variation of the magnetic elements of the earth. G. F. S.

QUANTUM MECHANICS.

BY

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THE concept of the atom, which considers the electrons moving in orbits restricted by certain quantum rules, has been, and still is, of extreme importance for the description of atomic properties. Many spectroscopic experiences and various other properties can be expressed in a very simple way with the help of this model.

The discovery and analysis of atomic properties, such as the Compton and photoelectric effects which did not fit into this orbital picture, were not however the only reasons which necessitated the development of the new Mechanics. A close study of the description of spectroscopic laws with the help of the atomic model reveals many unsatisfactory deductions, ad-hoc hypotheses and unnecessary complications. The introduction of "half" quantum numbers, the replacement of the square of some quantum number, say j^2, by $j(j+1)$, the existence of orbits with an angular momentum which has to be taken equal to zero, are some examples of this.

The new Quantum Mechanics, though allowing no longer a naïve picture for the atom, has made the great accomplishment of providing us with a completely consistent description of the atomic properties, giving at once the right results wherever the calculation has been possible. It is obvious that such a new theory changes considerably our conceptions of atoms and electrons, though it has been shown at the same time that the orbital picture is a very good approximation to the truth in most cases and the Quantum Mechanics corroborates even some of its results. For instance, it has been very surprising that the old theory gave such excellent results for the spectroscopic properties of alkali atoms, if one replaces the inner electron shell merely by a spherically symmetric charge distribution. This approximation has been completely justified by the Quantum Mechanics.

The introduction of the Theory of Groups into the Quantum Mechanics by Professor Weyl has thrown light upon a special class of atomic properties, namely those connected with symmetry. For instance, the selection rule for complicated spectra, the Laporte rule, which is so important for spectroscopy, has been shown by the methods of Weyl to be closely connected with the fundamentals of the Quantum Mechanics, whereas it did not find its right place in the older conceptions.

The success of this development gives hope that very soon some properties of spectra which still lack a rigorous explanation, may be described with the help of the Quantum Mechanics. Such problems of special interest for spectroscopy are, among others, the interaction between two spins, the interaction between the core of electrons and the magnetic properties of the nucleus. A rigorous solution of these problems will throw light on the laws of multiplet structure and on the properties of hyperfine structure.

HARMONIC ANALYSIS AND THE QUANTUM THEORY.

BY

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THE author desires to open the present paper with an apology. It purports to be not so much a new formal development in the quantum theory as an interpretation of results already obtained and a bridge connecting them with older branches of physics. Mathematical attacks on the quantum theory, except from the side of the professional physicist, have proved disappointing so far, and indeed Professor Weyl's treatment of the group theory is the only contribution from the side of the pure mathematician which has definitely become incorporated into the physics of quantum theory. It therefore seems highly doubtful whether the present paper will lead to any results important to the physicists. On the other hand, there is a large body of results belonging to the older physics which has up to the present time remained formally unconnected with the quantum theory, although its subject matter is the same and its results will have to be brought into connection with present work. Thus the pure mathematician may serve a temporary need, although the final version of the theory may altogether dispense with his help.

One of the remarkable things about light is that the quantities which are primarily introduced in the Maxwell theory are not observed; at any rate, in the case of moderately high frequencies. The Maxwell equations concern the electric and the magnetic vector. Our optical observations deal with the blackening of a photographic plate or a measurement by a photometer, or the determination of the intensity of light with a photoelectric cell, or with the estimation of this intensity by the naked eye, or other measurements of the same general sort. Every optical observation terminates in this manner. The quantities in the Maxwell theory which most nearly correspond with these observations are the

energy density, the Poynting vector, and similar expressions, which depend quadratically on electric and magnetic vectors. We thus find in classical optics the same relation between the linearity of the fundamental quantities entering into the equations and the quadratic character of observable quantities which plays such an important role in the quantum theory. In the latter the Schrödinger $\psi\bar{\psi}$ depends quadratically on the function which satisfies the Schrödinger equation. Similarly, in the revised theory of Dirac the quantities which are given a real physical existence are components of the electric current-density vector which again depends quadratically on the quantities satisfying the linear differential equations of Dirac. Indeed we do not need to go to such remote fields to find this contrast of linearity and quadraticity playing such an important role. If we consider the instruments which we use to measure alternating currents, voltages, power, etc., we see that in almost no case do they yield a linear reading in terms of current and voltage. The electrodynamometer records the attraction of two coils through each of which a current flows, and gives a reading proportional to the product of these currents. The hot-wire ammeter records the energy loss in a heated wire, and this again is proportional to the square of the current. As a matter of fact, the only instruments which give alternating-currens readings which are linear in current or voltage are the variout forms of oscillograph. These carry us a certain distance into the Hertzian region, but even they fail for the shorter Hertzian waves.

In order to understand the meaning of this linear-quadratic duality, it is very useful to see if we can find somewhere in mathematics a model of a similar duality. This we can find in connection with harmonic analysis. The development of a function in a Fourier series is a linear problem, as the coefficients depend linearly on the function expanded. The same thing applies to the expansion of a function as a Fourier integral. On the other hand, there are several important applications of harmonic analysis in which the coefficients of our trigonometrical development, which are complex quantities, possessing a phase as well as an amplitude, are of interest to us only in so far as the absolute value of their amplitude

is concerned. For example, when we break up light by a prism, we know nothing about the phase of the different rays into which the light is split. All that interests us is the absolute value of the amplitude of each component, or indeed the square of this absolute value, which is proportional to the power going into light of that frequency. Another illustration is the periodogram of Sir Arthur Schuster. If we have a phenomenon of Nature which we suspect of containing hidden periods, Schuster's method enables us to determine the squares of the amplitudes of the harmonic terms with the given periods. Again we discard phase. This is not due to the fact that we adopt in these cases an artificial system of analysis; it belongs to the nature of harmonic analysis itself.

Let us start with a function $f(t)$ which we wish to analyze trigonometrically. In the particular case where f is a sum of a finite number of complex exponential functions,

$$f(t) = \sum a_n e^{\Omega_n t}$$

we can associate with f a related function whose harmonic analysis is derivable from that of f in a very simple manner. Let us form

$$\varphi(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t + \tau) \bar{f}(t) dt.$$

That is, we take the conjugate imaginary function of $f(t)$, multiply it by f of t plus a displacement term τ, and average over all time; i.e. all values of t. This function will be itself the sum of a finite number of complex exponentials, with frequencies the same as those occurring in the harmonic analysis of $f(t)$, but with coefficients which are the squares of the absolute values of the coefficients in f. Thus harmonic analysis of this new function φ will contain all the information concerning f that is yielded by the study of frequencies and amplitudes, but no information concerning phases. If we then submit this function φ to an harmonic analysis we obtain what is substantially the periodogram or spectrum of f.

It is clear that φ is quadratic in comparison with f. What I propose to do in the present paper is to show how the quadratic character of this function is formally related

to the quadraticness of the quantum theory. Of course we need not confine our attention to functions f which are trigonometrical polynomials. We shall indeed consider all functions f for which the definition of φ, which we have given above, yields a value of φ for every τ. The harmonic analysis of φ is then carried out in accordance with the following definition:

$$r(u) = \int_{-\infty}^{\infty} \varphi(\tau) \frac{(e^{iu\tau} - e^{-\tau^2})d\tau}{i\tau}.$$

The function $r(u)$ which we thus obtain may be proved to exist, with an appropriate definition of the infinite integral, and to be definable as a monotone increasing function. $r(u)$ represents the total energy in the spectrum up to frequency "u." In case $r(u)$ has a derivative for a given value of u, $r'(u)$ represents the spectral intensity for this frequency.

It is not enough for the quantum theory for us to develop the spectrum theory of a single function. Our function $f(t)$ is replaced by a function $f(t, n)$, where n represents a parameter which may assume a discrete number of values, or a set of values corresponding to the points in an m-dimensional space, or finally, a set of values corresponding to these points, each with index assuming a discrete set of values. We can proceed to the formation of a function $\varphi(\tau, n)$ and a function $r(u, n)$, just as in the case of a single function. So far there is nothing of new interest. The essential difference between the spectrum theory with a parameter and the spectrum without a parameter only appears when the function $f(t, u)$ is subjected to a linear transformation. The φ and r corresponding to the transformed function cannot then be expressed exclusively in terms of the original φ and r. We are thus led to introduce the new set of φ functions:

$$\varphi(\tau; m, n) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} f(t + \tau, m)\bar{f}(t, n)dt,$$

and the corresponding r functions:

$$r(u; m, n) = \int_{-\infty}^{\infty} \varphi(\tau; m, n) \frac{e^{iu\tau} - e^{-\tau^2}}{i\tau} d\tau.$$

These functions depend upon two independent parameters,

and form what is known as a *matrix* in a finite or infinite number of rows and columns. They are in general complex unless $m = n$. It is a most important fact, and one which is easily proved, that the matrix of the quantities $r(u;\ m,\ n)$ is Hermitian or that

$$r(u;\ m,\ n) = \bar{r}(u;\ n,\ m).$$

We term this matrix a *coherency-matrix*. If we make use of this matrix as representing the spectrum of the set of functions $f(t,\ n)$, we see that a linear transformation no longer changes our spectrum in such a way that it cannot be expressed in terms of the untransformed spectrum. Indeed, if

$$g(t,\ m) = \sum a_{mn} f(t,\ n)$$

and $s(u;\ m,\ n)$ is the analogue of $r(u;\ m,\ n)$ when g is substituted for f,

$$s(u;\ m,\ n) = \sum\sum a_{mp} \bar{a}_{nq} r(u;\ p,\ g).$$

It thus appears that the effect of a linear transformation on the variables $f(t,\ n)$ is to multiply the matrix of the r's on the left by the adjoint of the matrix of this transformation and on the right by the matrix of this transformation itself. This transformation of our matrix has a physical meaning. If we take an optical instrument, its effect is to yield at the plane of the image an electromagnetic disturbance which depends linearly on the electromagnetic disturbance at the object plane. Our most general observation of the image plane now would yield us point by point the spectrum of the light in this plane. (This is a somewhat simplified version of what happens, but is not false in character.) The optical instrument then transforms the matrix representing the electric disturbance at the object plane linearly into the matrix representing the electric disturbance at the image plane, and then yields us the spectrum of this, which consists in the diagonal terms of the transformed matrix.

This stands in exceedingly close relation to the procedure of quantum mechanics. We find an Hermitian matrix exactly analogous to that of quantum mechanics. This matrix is subject to linear transformations in both cases, but once it has been transformed, only the diagonal terms have a physical

meaning. The transformations in each case correspond to the use of definite physical instruments. Once the diagonal terms have been formed, it is no longer true that, if we subject them, or the matrix made up of them alone, to a new linear transformation, we shall get the same matrix or the same diagonal terms which we would have got from our original matrix by the two transformations repeated end on end, as in an intermediate stage we have discarded all non-diagonal terms. This corresponds to the fact often cited in the quantum theory that an instrument always distorts and modifies the atomic phenomenon which it is meant to measure.

The transformations corresponding to the optical instruments at our disposal have the mathematical property of forming a group. If we examine the image produced by one system of lenses through another system of lenses, the combined systems form themselves an optical instrument. This group will in general be more restricted than the group of all possible linear transformations. In the case of light, for example, no optical instrument can transform a solution of Maxwell's equations into anything but a solution of Maxwell's equations. What the precise group of transformations will be in any particular case requires an investigation in that particular case, but in very many instances we shall find some Hermitian expression in the coefficients of the linear transformation invariant, and this invariant will restrict our group to the sort that Weyl calls a group of Unitärtransformationen. It is worth while pointing out, however, that the origin of the group theory is from transformations which may or may not be unitary, and that Weyl's restriction becomes difficult to verify or even to formulate under the assumptions of general relativity.

It is perhaps worth while to give working models of the ideas involved in this type of harmonic analysis. The matrix of the r's plays an important rôle in the statistical analysis of time series, and is indeed the closest analogue for statistical series of the notion of correlation matrix among frequency series. If we have a number of different quantities $x_1^{(n)}$, $x_2^{(n)}$, $x_3^{(n)} \cdots x_m^{(n)}$ corresponding to a single occurrence n, the correlation matrix of these quantities is the one whose terms are:

$$\sum x_i^{(n)} x_j^{(n)}.$$

If now we plot the points $(x_1^{(n)}, \cdots, x_m^{(n)})$ in an m space, as we can do when m is 2, we get a figure known as a scatter-diagram. One interesting example of a scatter-diagram is that in Professor J. L. Coolidge's book on statistics in which he plots in such a manner the Big Bertha shells that fell on Paris. Of the linear transformations to which the coördinates x and y in this diagram are subject the most natural are obviously those consisting of translation and rotation; that is, the orthogonal. Now we can connect with any such diagram an ellipse, which best represents the points of the diagram. The problem of doing this is the same as the problem of reducing the matrix to diagonal form by an orthogonal transformation. This ellipse, that is, will yield us a diagonal matrix if we refer our scatter-diagram to its axes as coördinate axes. It is thus connected with the diagram in a manner which remains invariant under the rotation group.

On the other hand, scatter-diagrams may be used for all sorts of statistical problems where the quantities with which we are dealing have no sort of natural scale of measurement, and where rotation is of no importance whatever. The group of transformations which is significant in such a case is the affine group preserving horizontal and vertical directions. The quantities which interest the statistician are thus invariants of the scatter-diagram with respect to this group. Without going any further it may be said that correlation coefficients and lines of regression are such invariants.

To come to an example more directly allied with quantum theory, let us look into the coherency-matrix representation of polarized light. We here consider two components of the electric vector $(x(t), y(t))$ as functions of the time t. The coherency-matrix may be represented as the sum of the four following matrices each multiplied by an appropriate real coefficient.

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \qquad \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \qquad \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \qquad \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix}$$

1 2 3 4

The coefficient of matrix 1 represents the total intensity of

the light in question. The coefficient of matrix 2 represents the excess of light polarized in a horizontal direction over that polarized in a vertical direction, let us say. The coefficient of matrix 3 similarly represents the excess of light polarized at 45 degrees over light polarized at 135 degrees. The coefficient of matrix 4 represents the excess of light polarized circularly clockwise over light polarized circularly in a counterclockwise direction. It will be noted that in the form given matrices 1 and 2 are diagonal. Any unitary transformation leaves 1 unchanged. On the other hand, the matrices 3 and 4 are now free of diagonal terms. The unitary transformation which changes 2 into 3 will transform 3 and 4 into matrices clear of diagonal terms and a similar remark applies to the transformation which transforms 2 into 4. Physically the significance of these comments is the following: There are four optical instruments which together are sufficient to give us all possible information concerning the polarization of a beam of light. The first is a photometer; the second is obtained by combining a photometer with a doubly-refracting crystal in a certain plane orientation and reading the difference in the intensities of the ordinary and extraordinary rays; the third is obtained by rotating this instrument through 45 degrees; and the fourth measures the degree of circular polarization of the light in a similar manner. The light emerging from each of the instruments 2, 3, and 4 appears to each other one of the instruments as if it were unpolarized. The instruments of 2, 3, and 4 determine a group, which has already been discussed by Weyl in connection with the Pauli spinning electron. Clearly in this case the significant group is a sub-group of the unitary group.

It may now be asked, "What has all this to do with quantum theory? How can we interpret the matrices of quantum theory as coherency matrices?" Two alternatives suggest themselves. In the first, the coherency is taken on the fifth unobservable direction of Kaluza, Klein, and Fock. In the second, the coherency is taken on the time. The relativist invariance of the first form of the theory is obvious. The functions whose coherency is taken are solutions of the five-dimensional Schrödinger equation:

$$\Delta\Psi = 0. \qquad (1)$$

The coherency matrix of a solution of this equation is taken for one specific x frequency; namely, $h/2\pi$. The problem of reducing this matrix to diagonal form is then the problem of determining the characteristic functions of the Schrödinger equation. It is indeed true that there is no immediate reason why just precisely the characteristic functions with respect to the time should bring this matrix into diagonal form, but if we remember that a single solution of the equation is unobservable to us, and that all we ever can do is to observe the sum of a large number of coherency matrices which are placed at haphazard with respect to the time, it will be seen that the real positive terms on the principal diagonal will ultimately become very large in comparison with the terms away from the principal diagonal, which are distributed at haphazard in phase. Thus if

$$f(x, y, z, t) = \sum a_n e^{i\lambda_n t} \psi_n(x, y, z),$$

where the ψ_n's are the characteristic functions and the λ_n's the characteristic numbers of the Schrödinger equation, the expression

$$\sum |a_n|^2 \psi_n(t; x, y, z) \bar{\psi}_n(t; x, y, z)$$

represents everything that can be determined from the coherency matrix of the function f. Consequently we see that the Schrödinger theory picks on just the one aspect of the function f which can have physical significance.

Of course the Schrödinger theory in its original form has been supplanted by the Dirac modification. How this Dirac modification can be put into analogous shape depends upon connecting it with a five-dimensional equation analogous to (1). So far, attempts in this direction have met only with moderate success, but it is perhaps too early to say that they are hopeless.

The other alternative method of approach is to make our coherency on the time dimension. This is of course not invariant under relativist transformation, at least in appearance. If, however, we take as our wave equation the ordinary equation in free space, we see that our coherency matrix when reduced to diagonal form yields us just the same

quadratic distribution over all possible plane waves with velocity 1, which we should obtain if we were to extend our spectrum theory into four dimensions and form this four-dimensional spectrum of the solution of the wave equation. As I hint here, it is possible to extend the spectrum theory which I mentioned earlier in the article to cover the case of spectra in time and space simultaneously. This suggests an alternative on which I am at present at work, but which is not yet sufficiently mature to be presented in detail. It is possible to extend this spectrum theory even further and to analyze in terms of a set of functions more general than the complex exponential functions:

$$e^{i(x\xi+y\eta+z\zeta+t\tau)}.$$

This analysis may be carried through in a completely invariantive way. It is probably this invariantive spectrum theory which is to be taken as the true substratum of the coherency-matrix theory, for in the case where the functions in terms of which the analysis is made are trigonometrical in the time, the result of this three-dimensional-spectrum analysis coincides precisely with the array in space Eigenfunktionen and time frequencies which we obtain by reducing to diagonal form the time coherency matrix of the function analyzed.

HARMONIC ANALYSIS AND THE QUANTUM THEORY.

DISCUSSION BY

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I SHOULD like to make a few specific remarks concerning Professor Wiener's interesting paper and the subject—the place of Harmonic Analysis in quantum theory—with which it deals. In the first place, Professor Wiener has shown how one can obtain coherency functions or matrices from solutions $f(x, y, z, t)$ of Schrödinger's equation by taking the coherency with respect to the time t or the Kaluza-Klein fifth dimension, considering the remaining variables as parameters or indices of a three or four parameter family of functions. But we could also take the coherency with respect to any or all of these remaining variables by repeating the process an appropriate number of times. I shall briefly indicate the result of taking the coherency with respect to all four variables occurring in Schrödinger's function and give its quantum-mechanical interpretation for the case of free motion of a particle.

I first associate with f the function

$$\varphi(\xi, \eta, \zeta, \tau) = \lim_{X \to \infty} \frac{1}{2X} \int_{-X}^{+X} dx \cdots \lim_{T \to \infty} \int_{-T}^{+T} f(x + \xi, \cdots, t + \tau) \bar{f}(x, \cdots, t) dt$$

(assuming here and in the following that f is so constituted that the various processes involved are possible and have a meaning) by averaging over each of the four variables in turn. If f be the sum of complex exponential functions φ will also be such a sum, but its coefficients will be the squares of the absolute values of the corresponding coefficients of f. Taking the coherency of $\varphi(\xi, \eta, \zeta, \tau)$ with respect to τ we have

$$r(\xi, \eta, \zeta : \lambda') = \int_{-\infty}^{+\infty} \varphi(\xi, \eta, \zeta, \tau) \frac{(e^{-\Omega\tau} - e^{-\tau})^2}{i\tau} d\tau.$$

But I shall deal with the difference

$$2\pi r(\xi, \eta, \zeta : \lambda'; \lambda'') = r(\xi, \eta, \zeta : \lambda'') - r(\xi, \eta, \zeta : \lambda')$$

of two such r's instead of with the original r itself, i.e., in Professor Wiener's terminology, with the energy between the two frequencies λ', λ'' instead of with the total energy up to frequency λ''. I then perform the analogous process for each of the variables ξ, η, ζ in turn by introducing variables α, β, γ in place of them in the same way that τ was replaced by λ; the result is a coherency function $r(\alpha', \beta', \gamma', \lambda'; \alpha'', \beta'', \gamma'', \lambda'')$ depending on eight parameters. A set of functions f_n would in the same way give rise to an Hermitean coherency matrix $\|r_{mn}\|$. On applying this to the function

$$f(x, y, z, t) = \sum a(\alpha, \beta, \gamma, \lambda) e^{i(\alpha x + \beta y + \gamma z + \lambda t)},$$

where the summation is taken over a range of values of the parameters α, β, γ, λ (integration if the range is continuous) we find

$$r(\alpha', \beta', \gamma', \lambda'; \alpha'', \beta'', \gamma'', \lambda'') = \sum |a(\alpha, \beta, \gamma, \lambda)|^2,$$

where the sum (or integral) is taken over all values of α, etc., between α', α'', etc.

Consider now the case in which such an f is a solution of Schrödinger's equation

$$\nabla^2 f + \frac{4\pi i \mu}{h} \frac{\partial f}{\partial t} = 0.$$

The parameters are then subject to the restriction

$$\lambda = -\frac{h}{4\pi\mu}(\alpha^2 + \beta^2 + \gamma^2) \qquad \left(= -\frac{2\pi}{h} E \right).$$

The term of f having these parameters represents the de Broglie wave for a free particle with total energy $-\lambda h/2\pi$ and direction determined by the ratios $\alpha : \beta : \gamma$. The (normalized) amplitude $a(\alpha, \beta, \gamma, \lambda)$ is interpreted as the probability that the particle will be in this state. Our coherency function $r(\alpha', \beta', \gamma', \lambda'; \alpha'', \beta'', \gamma'', \lambda'')$ is consequently the probability that the particle have energy E lying between $-\lambda'' h/2\pi$ and $-\lambda' h/2\pi$ and direction $\alpha : \beta : \gamma$

between $\alpha' : \beta' : \gamma'$ and $\alpha'' : \beta'' : \gamma''$. It seems likely that the extension of these ideas to other sets of orthogonal functions, as suggested by Professor Wiener in the latter part of his paper, would serve an analogous purpose for general quantum-mechanical systems.

I should also like to call attention to the connection between Professor Condon's discussion of momentum space and the theory of harmonic analysis. Expressed in the terminology employed by Professor Weyl in his paper, Professor Condon introduces in the unitary system-space $\mathfrak{R}$ a coördinate system in which the operator $\mathbf{p}$ representing the momentum is referred to principal axes, in which case the operator $\mathbf{q}$ may often be represented by $-\frac{h}{2\pi i}\frac{\partial}{\partial p}$. This coördinate system must be related to that used by Schrödinger (in which the principal axes of the operator $\mathbf{q}$ are taken as coördinate axes) by a unitary transformation of $\mathfrak{R}$. It is in fact readily shown—by finding the principal axes of the operator $\frac{h}{2\pi i}\frac{\partial}{\partial q}$ in Schrödinger's coördinate system—that the components $\psi(p)$ of the vector Ψ defining the state in the former coördinate system are related to its components $\psi(q)$ in the latter by the transformation

$$\psi(p) \sim \int e^{-(2\pi i/h)pq}\psi(q)dq$$

and that the operator $\mathbf{q}$ becomes $-\frac{h}{2\pi i}\frac{\partial}{\partial p}$. In other words —and this is the connection to which I wish to call attention —the function $\psi(p)$ is the Fourier transform of $\psi(q)$. In such cases the equation obtained by Professor Condon for $\psi(p)$ is the equation of the Fourier transform of $\psi(q)$. Darwin has in fact employed the inverse of this transformation in the quantum-mechanical discussion of certain simple types of motion.

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The Effects of War Gases. The Chemical Warfare School of the Edgewood Arsenal, Maryland, has prepared a pamphlet comprising a comparative study of gas and other weapons in the recent war. The pamphlet was prepared by Colonel H. L. Gilchrist, U. S. Army, Chief of the Medical Division, Chemical Warfare Service, issued from the Government Printing Office.

The question of the permissibility of poisonous gases in war was discussed many years ago without any definite decision. In the Hague Convention of 1899 action against the use of such gases was proposed. It was pointed out, however, that all explosives used in shells produce more or less dangerous products and it was suggested that action be taken prohibiting the use of any apparatus for the sole purpose of distributing asphyxiating gases. This agreement was signed by all the nations represented except the United States and Great Britain. At the convention of 1907 Great Britain signed, so that when the World War broke out the United States was the only nation of any importance that had refused to prohibit this method of warfare. The Hague Convention provides that if a belligerent takes an ally not signatory to any agreement the opposing parties are released.

When the United States went into the war Germany would have been permitted to use poisonous gases but Germany anticipated this with some very serious results to the allied forces. It can be shown, that it may have been fortunate that the Germans did anticipate in this matter, but this phase need not be discussed here. The pamphlet above noted gives a large amount of valuable information concerning the effects of poisonous gases as compared with other methods of attack. It seems clear that, notwithstanding all that has been said about these gases they represent a less disastrous ultimate effect than the other methods of warfare. Unfortunately the effects have been made responsible for a large amount of illness in subsequent years with which, in reality, the gassing had no concern. In the very large majority of cases the diseases which have appeared in men gassed during the war are due either to the natural decline of health in advancing years, improper habits of life, or causes commonly operative in the transmission of disease.

H. L.

THE KINETIC METHOD IN STATISTICAL MECHANICS.

BY

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So much has been said to-day about the statistical interpretation of quantum-mechanics, that perhaps one should like to see now quite clearly, how the fundamental postulates of the ordinary kinetic theory and statistical mechanics,[1] especially their connection with thermodynamics, are "explained" by the quantum theory. In principle this is nowadays certainly possible,[2] I think, though I am not able to show you exactly how. Therefore only a few remarks, especially on the so-called kinetic method.

We can distinguish in the kinetic theory *two* fundamental methods:

a. The statistical method; the equilibrium state is described as the most probable or as the mean state. The range of application is about the same as of ordinary thermodynamics,[3] because it deals principally with equilibrium states, with the exception [4] that it can explain also the *fluctuation-phenomena* around an equilibrium state.

b. The kinetic method or *method of collisions.* Here we try to explain the macroscopic properties by considering more precisely *the interaction* between the particles (atoms, light quanta, etc.), out of which our system is composed. It is the only method, for systems *not* in an equilibrium state, for finding the *rate* of physical and chemical processes. Especially it gives, in principle, by means of the H-theorem of Boltzmann, something about the velocity with which the equilibrium state is reached. A disadvantage is, that it

[1] Compare for instance my dissertation, Over statistische methoden in de theorie der quanta (The Hague, Nyhoff, 1927).

[2] Compare J. v. Neumann, *Gött. Nachr.*, 1927, p. 273.

[3] Partly, it is simply another language.

[4] This is, perhaps, not exactly true, in the sense that by a slight reformulation of the second law of thermodynamics one can also account for the fluctuation-phenomena. Compare L. Szilard, *ZS. f. Phys.*, v. 32, p. 753, 1925.

cannot account for the fluctuation-phenomena, at least not in a general way.

I think that quantum-mechanics will combine these two, now almost distinct, methods by sharpening the kinetic method, using statistical considerations.

In the kinetic method we can deduce, at least formally, all the thermodynamic laws by the following assumptions: [5]

1. The "Stoszzahlansatz": the number of collisions per sec. of particles in the quantum state i with particles in the state j, where k and l are the states after collision, is:

$$A_{ij}^{kl} = a_{ij}^{kl} n_i n_j (1 + \theta n_k)(1 + \theta n_l).$$

a_{ij}^{kl} is a sort of transition probability, and

$\theta = -1$ for a Fermi-Dirac-gas,

$\theta = 0$ in the classical case,

$\theta = +1$ for a Bose-Einstein-gas.

2. The principle of microscopic reversibility: [6]

$$a_{ij}^{kl} = a_{kl}^{ij}.$$

Introducing a generalized H-function of Boltzmann by [7]

$$H = \sum_i \{n_i \log n_i - \theta(1 + \theta n_i) \log (1 + \theta n_i)\},$$

we can show that $(dH/dt) < 0$ until the equilibrium-state is reached, which is determined by the *functional equation*

$$\bar{n}_i n_j (1 + \theta n_k)(1 + \theta \bar{n}_l) = \bar{n}_k n_l (1 + \theta \bar{n}_i)(1 + \theta \bar{n}_j),$$

where the bar denotes the values in the equilibrium state. From this follows immediately, when we take into account only the conservation of energy during each collision (no translational motion of the whole gas):

$$\log \frac{n_i}{1 + \theta n_i} = -\beta \epsilon_i + \log A$$

[5] This is for a single gas. Compare P. Jordan, *ZS. f. Phys.*, v. 41, p. 711, 1927; L. S. Ornstein and H. A. Kramers, *ZS. f. Phys.*, v. 42, p. 481, 1927; L. Nordheim, *Proc. Roy. Soc.*, series A., v. 119, p. 689, 1928.

[6] See the discussion in R. C. Tolman, Statistical Mechanics (New York, The Chemical Catalog Company, 1927), chapter 15.

[7] The entropy is always given by $S = -kH$.

or

$$n = \frac{1}{\frac{1}{A} e^{\beta\epsilon_i} - \theta}.$$

It is easy to generalize this for mixtures, to find the analogue of the transport equation of Boltzmann, and so on.[8]

I wanted further only to point out that, in the case of the interaction of light and fixed atoms, all these formal assumptions are beautifully explained by the theory of Dirac.[9] Especially the principle of microscopic reversibility is there "proved" by showing that $a_{ij}{}^{kl}$ is equal to the square of the absolute value of an Hermitean matrix.

With the help of this we can see also more precisely how fast the equilibrium state (Planck's radiation formula) is established. As is well known, Planck always mentioned that we need "a little piece of coal dust" to establish the equilibrium, when we have the radiation in a vessel, of which the walls are perfect mirrors. We may ask now, how big this piece of coal dust has to be?

First it is clear that we need some matter, because light waves or light quanta have no way to interact, will therefore not "collide," or, we may say, are perfectly penetrable for each other. Let us therefore take first one free electron. One sees easily that, when we consider only the Compton-scattering, this will not establish the equilibrium.[10] Because then, also, the *number* of light quanta is conserved (besides their energy), and from the statistical treatment it follows that we can only get

$$\overline{n}_i = \frac{1}{\frac{1}{A} e^{\beta\epsilon_i} - 1}$$

and not Planck's formula, for which $A = 1$. We have to

[8] See L. Nordheim, *Proc. Roy. Soc.*, series A., v. 119, p. 689, 1928.

[9] P. A. M. Dirac, *Proc. Roy. Soc.*, v. 114, p. 243, 1927; v. 114, p. 710, 1927. Compare also the treatment in H. Weyl: Gruppentheorie und Quantenmechanik (Leipzig, S. Hirzel, 1928), § 20 and 44. For the extension to an electron gas, see esp. P. Jordan and E. Wigner, *ZS. f. Phys.*, v. 47, p. 631, 1928.

[10] P. Jordan, *ZS. f. Phys.*, v. 33, p. 649, 1925. W. Bothe, *ZS. f. Phys.*, v. 23, p. 214, 1924.

determine A from the condition that

$$\sum_i \bar{n}_i = N.$$

The function $N(A)$ will be zero for $A = 0$, and infinite for $A = 1$, and is always curved toward the A axis. It is clear, that the electron will *very slowly* make N infinite, namely by triple and higher collisions. It is therefore not a very effective "piece of coal dust." Neither is one Hydrogen-atom, as we see from Dirac's treatment, because that can only establish Planck's law for its own discrete frequencies; at least its effectiveness for other frequencies must be very slight, since here again collisions of higher order are necessary. So that we may say, perhaps, that we really need a piece of crystal, or at least some material system with a practically continuous energy spectrum, to establish the equilibrium *rapidly*.

In conclusion, I may say, perhaps, that it seems to me better now to stress more the *difference* between light quanta and electrons, than their similarity. I have the impression that light is still more "wavy" than electrons, for the following reasons:

a. For light quanta holds the Bose-Einstein statistics which is equivalent to quantized eigenvibrations; for electrons we have the Fermi-Dirac statistics, which cannot be translated in a natural way into a three-dimensional wave language.

b. To find the coupling terms between light and matter we use always the classical expressions following from the electro-magnetic *wave*-theory of light and from the *corpuscular* theory of matter (esp. Coulomb's law).

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NOTES FROM THE U. S. BUREAU OF STANDARDS.*

CHROMIUM PLATED LABORATORY WEIGHTS.

THE hardness of chromium plating is a considerable advantage in the case of weights that are much used, but this advantage is somewhat offset by the susceptibility of such a coating to attack by fumes of hydrochloric acid. However, it seemed worth while for the bureau to try this plating on three sets of weights which have been put into use in the chemistry division. In preparing these sets it was found that great care is needed in washing the plating liquid out from under the knob. If this is not done the weights vary with changes in atmospheric humidity, as is the case with gold-plated weights. Some indication of the degree of care needed was given by the fact that one of the three sets was at first too variable to warrant regular certification, although a careful laboratory assistant, familiar with electroplating work, thought that he had washed this set thoroughly, and just as carefully as the other two sets.

As in the case of gold-plated sets, it was found that this chromium-plated set could readily be made constant by proper boiling and rinsing.

QUALITY OF PURIFIED WOOD FIBERS.

THE following is a summary of a report on purified wood fibers presented at the annual meeting of the Technical Association of the Pulp and Paper Industry, held in New York on February 19.

As a result of an investigation in progress at the bureau, it is indicated that certain types of highly-purified wood fibers, which are commercially obtainable at the present time, are suitable for conversion into high grade bond and permanent record papers, as well as any other types of papers where the qualities of durability and permanence are essential.

* Communicated by the Director.

Such papers have hitherto been made only from the best grades of rag fibers, which are considerably more expensive than the purified wood fibers.

Pure cellulose, commonly called alpha cellulose, has a high degree of permanency. The cotton fiber is the purest form of cellulose found in nature. For this reason papers carefully prepared from high grade cotton rags have hitherto been used exclusively where permanence extending over hundreds of years is desired. The ordinary wood fibers, even those of the best grades, have impurities and degraded forms of cellulose present which seriously affect their permanence. By a series of chemical treatments, which have been developed by a pulp and paper manufacturer, these objectionable impurities are removed, leaving a fiber similar in its chemical composition to the cotton fiber and having the desired paper-making characteristics.

In order to find out how the purified wood fibers would endure as compared with other commonly-used paper-making fibers, tests are being made of the various types of paper-making wood fibers and of several grades of rag fibers. These tests include chemical purity, whiteness, and microscopical structure. A thorough investigation of the strength and durability of a series of representative commercial papers prepared from fibers similar to the above is also being made.

The relative durability of the various paper-making fibers is studied by means of accelerated aging tests, whereby a few hours' treatment simulates the effect of natural aging over a long period of years. The samples are baked in dry form at 100° C., are cooked with steam, and are exposed to intense light rays from an artificial sun. After all these severe treatments, the samples are again subjected to searching tests to find how much they have deteriorated both chemically and physically. In all cases, while the ordinary wood fibers show the expected rapid deterioration, the purified wood fibers withstand the aging tests practically as well as the highest grade of rag fibers.

While further study is being made, it would appear that fibers approaching closely to cotton fiber in paper-making value can be made from materials having originally much less value than cotton. This development is similar to that

of rayon and, in fact, is an off-shoot from a study of the utilization of wood fiber for making rayon.

CORNSTALKS FOR PAPER MAKING.

The manufacture of paper from cornstalks is a matter of great interest to the Bureau of Standards, as an important phase of the general problem of utilization of waste products from the land.

For the past two years, the bureau has been coöperating with Iowa State College in producing a cornstalk insulating board, on a semicommercial scale. While the pulp used in making this board is of a different quality from that required for paper, the work has yielded rather definite information on the cost of cornstalks and on the nature of the cornstalk as a source of fibrous material. In connection with this work, an economic survey was made, which indicates that cornstalk pulp may now be able to compete with wood pulp if it can be made of satisfactory quality.

Previous investigators have shown that paper made wholly of cornstalks is so hard and harsh as to be objectionable for many purposes. However, most papers now on the market are made of blended pulps, such as sulphite and ground wood, sulphite and rag, etc. It would seem that cornstalk pulp might be used as an ingredient in such a blend, and that its characteristic harshness may even be used to advantage in making a paper of different properties from any now on the market. To work out such a blend, when the experiments are made on a full commercial scale, is both laborious and expensive. The newsprint used in printing the Sunday Commercial News of Danville, Illinois, December 16, 1928, is the result of one such experiment. This paper contains 55 per cent. cornstalk pulp, which is evidently too much, for the paper is too harsh to be satisfactory as newsprint.

Another well-known characteristic of cornstalk pulp is its "dirtiness." The cornstalk contains several kinds of fibers, which offer different degrees of resistance to being reduced to pulp. If the stalks are cooked but slightly, the more difficultly reduced fibers remain in a raw condition and show up as brown flecks in the paper. If the stalks are cooked hard,

the more easily reduced fibers are gelatinized, and the drying of these spots of jelly produce "shiners" in the papers. The paper used in printing the book "Farm Wastes in Industry" by G. M. Rommel, contains 25 per cent. of cornstalk pulp, and shows these flecks quite plainly. While they are not particularly objectionable in book paper, pulp containing them could not be used at all in making writing paper.

There are thus two technical problems to be solved in making cornstalk paper; finding out how to cook cornstalks to make a clean pulp, and finding out how to blend cornstalk pulp with other pulps to make papers of the desired qualities.

Private capital is making an attempt to solve these problems. In view of the obvious importance of the project to both the farmers and the paper industry, it is believed that the Government is justified in coöperating with private industry in giving such assistance as it can in the solution of these two technical problems. Work on the project has already been started at the bureau, and will be continued intensively during the next fiscal year.

Since both of these technical problems will affect the cost and selling price of cornstalk paper, the economic problem must await the development of further information.

STRENGTH OF SAND-LIME BRICK.

THE bureau, in the course of its investigation of the properties of sand-lime brick, has accumulated considerable data on the strength of the brick. However, the majority of tests of the commercial product have been made on small lots, usually five or ten brick from each manufacturer, as required by the specifications. It was considered desirable to obtain additional information on the strength of the brick, and particularly on the variation in strength of the product produced by an individual manufacturer. Consequently, 25 manufacturers have supplied 50 brick each, representing their regular production, for use in these tests.

After drying to constant weight, the absorption by total immersion in cold water was obtained at the five minute, five hour, and 24 hour periods. The absorption on boiling for five hours was also determined. Ten of each group of 50

brick were then selected, using care to obtain a range of absorption, and these ten were reserved for freezing tests, to determine the weathering properties of the brick. The other 40 were dried, and broken transversely to determine the modulus of rupture. The compressive strength of each half of each brick was then determined, one-half having been tested on edge, the other half tested flat. The results obtained thus far show that the absorption ranged from 13.7 to 21.6 per cent. of the weight of the dry brick; the modulus of rupture from 330 to 730 lbs./in.^{2}; the compressive strength flat, from 2,150 to 4,470 lbs./in.^{2}; and the compressive strength on edge, from 1,500 to 3,850 lbs./in.^{2}

COMPRESSIVE AND TRANSVERSE STRENGTH OF BRICK.

THE April issue of the *Bureau of Standards Journal of Research* will contain a paper reporting the compressive strength flat and on edge, and the transverse strength of 27 makes of bricks covering a range of conditions in method of manufacture and degree of burning. The attempt is made to correlate the variation in ratios of these different measures of strength with the various structural features of the brick.

The following conclusions apply to the data thus presented:

1. The tendency of soft mud brick is to give higher unit strengths when tested on edge than when tested flat.

2. The "compacting effect" on the structure of the edge-set brick by the superimposed weight of the other bricks in the kiln is offered as a tentative explanation of the tendency toward higher strength on edge.

3. This tendency toward higher strength on edge is overcome by laminar and cracking structure in the case of certain bricks.

4. Soft mud brick tend to display less deviation in the ratio modulus of rupture to flat compressive strength than any of the other methods of manufacture, but even with these the ratio ranged between 0.13 and 0.26.

5. Auger lamination in end-cut brick appears to be associated with high ratios for modulus of rupture to flat compressive strength.

6. Die lamination in side-cut brick appears to be associated

with low ratios for modulus of rupture to flat compressive strength.

7. The ratio of the compressive strength of brick tested flat to the compressive strength of brick tested on edge ranges from 0.74 to 2.3.

8. In view of the variation in the ratio of the compressive strength of brick tested flat to the compressive strength when tested on edge, there exists no general rule for converting values from one kind of test to the other kind.

9. The ratio of the modulus of rupture to flat compressive strength ranged from 0.426 to 0.070.

10. In view of the variation in the ratio of the modulus of rupture to compressive strength when tested flat the possibility of inferring a compressive strength from a transverse test or vice versa is open to very large errors for any given make of brick.

CAST STONE.

THE bureau has determined the absorption of numerous samples of cast stone. In connection with this work it has been suggested that the absorption may be due to a low dry weight of the sample caused by partial dehydration of some of the colloidal material present in the stone when the latter is dried at temperatures as high as 110° C. It is also possible that high absorptions may result from boiling the sample for the total absorption, thus producing continued hydration of some of the cement which has not set.

With these possibilities in mind, the following method of absorption was tried. Cores were cut from each of twelve typical samples. These were maintained at 35° C. in dry air until a loss of not more than 0.05 per cent. in 4 hours was obtained. The specimens were then totally immersed in 300° mineral seal oil and the absorption noted over a period of 84 hours. Then the specimens and oil were placed in a closed chamber, the pressure reduced and maintained at 4 mm. mercury for 5 hours. The absorptions were then calculated to the equivalent of water as per cent. of dry weight of specimens. The specimens absorbed from 1.6 to 9.3 per cent. in 48 hours, one specimen being below 2 per cent. After reducing

the pressure to 4 mm. the absorptions ranged from 2.2 to 13.4 per cent.

Thus, even under conditions where the possibility of dehydration is reduced and the rehydration eliminated, cast stone does absorb a considerable quantity of water and the absorption as generally carried out, using water, is a fair measure of the pore space of the stone.

REFRACTORIES INVESTIGATION.

REFERENCES to data of interest which have developed as a result of the study of fundamental properties of fire-clays and fire-clay bricks have appeared from time to time in the Technical News Bulletin, the most recent reference having been made in the August, 1928, issue. A progress report, giving in detail the complete method of procedure followed in this study and the results of tests, have been prepared for publication at early date. A brief summary of the most important results is here given.

Thermal expansion data determined on the fire-bricks "as received" from the manufacturer and after reheating in the laboratory kilns at 1,400, 1,500, and 1600° C. show that both the rate of expansion and total expansion of the majority of bricks decrease as the temperature of firing is increased. These data also indicate that the thermal expansion behavior of the finished bricks may not necessarily be the average rates of expansion of the clays entering into their manufacture.

The modulus of elasticity of the bricks increased greatly at 550° C. when compared to values obtained in tests made at 20° C. At 1000° C. the modulus of elasticity decreased greatly, i.e., the bricks became less rigid and more subject to plastic deformation. The modulus of elasticity and transverse strength determined at room temperature increased as the temperature of firing of the bricks was raised. Resistance of fire-bricks to spalling decreased with increase of modulus of elasticity and increase of thermal expansion.

Data were obtained which show that sufficient differences exist in the physical properties of bricks of the same brand to cause large variations in the number of quenchings required to cause spalling.

An empirical relation of the modulus of rupture, transverse strength, coefficient of expansion, and total grog content of the brick is given, which results in a factor expressing the relative resistance of the various bricks to spalling in the water quenching test.

STANDARD SAMPLES.

A STANDARD sample of opal glass of the following composition is now ready for distribution: Silica 67.5 per cent.; calcium oxide 10.5 per cent.; sodium oxide 8.4 per cent.; potassium oxide 3.2 per cent.; alumina 6.0 per cent.; fluorine 5.7 per cent.; arsenic trioxide 0.2 per cent.; arsenic pentoxide 0.1 per cent.; and ferric oxide 0.08 per cent. The price of this glass is $2 per sample of 45 grams. If remittance does not accompany the order the samples will be sent by parcel post C. O. D. to addresses in the United States.

STANDARD VISCOSITY SAMPLES.

THE bureau has for sale several types of oil samples (6 petroleum and 1 castor oil) for use in calibrating viscometers, ranging in viscosity from 20 to 8,000 centipoises, between temperatures of 15 to 30° C. The charge is $5 per 1-quart sample, which includes a statement of the exact viscosity (within $\pm$ 0.5 per cent.) at any one temperature at the time of purchase. Additional viscosities will be charged for at the rate of $5 for temperature.

Remittance payable to the "Bureau of Standards" must accompany order. Samples will be forwarded via express collect to addresses in the United States. Orders from foreign countries should include an amount to cover parcel post for a total weight of 5 pounds including packing.

NOTES FROM THE RESEARCH LABORATORY, EASTMAN KODAK COMPANY.*

EFFECT OF ENVIRONMENT ON PHOTOGRAPHIC SENSITIVITY. I.[1]

S. E. Sheppard and E. P. Wightman.

THE reversible pH effect upon the speed of emulsions as observed by Rawling and Rawling and Glassett has been investigated. Aged plates, treated by bathing, give small and uncertain effects. Fresh emulsions appear to show the effect more definitely. Considerably more work is required before the nature of the effect can be definitely established. The possible causes appear to fall in four classes: (1) An action on silver bromide; (2) an action on silver nuclei; (3) an exaltation of a bromine acceptor function of the silver sulfide nuclei during exposure as suggested by K. C. D. Hickman; and (4) a reversible effect on gelatin.

PHOTOGRAPHIC CHARACTERISTICS OF MOTION PICTURE STUDIO LIGHT SOURCES.[2]

L. A. Jones and M. E. Russell.

THE fundamental relationships which determine the visual and photographic efficiencies of light sources are discussed briefly. A direct experimental method of measuring relative photographic efficiencies is then described. Numerous data are given relating to all of the light sources commonly used in the motion picture studio, including those introduced recently, such as the high efficiency Mazda lamps, white, orange, and yellow flame carbons, and the mercury vapor are in combination with a new gaseous conductor neon lamp. The

* Communicated by the director.

[1] Communication No. 349 from the Kodak Research Laboratories and published in *Phot. J.* **69**: 22 (1929).

[2] Communication No. 358 from the Kodak Research Laboratories and published in *Trans. Soc. Mot. Pict. Eng.* **12**: No. 34, 427; *Phot. Ind.* **26**: 1051–1077 (1928).

problem is considered both from the standpoint of relative photographic candle power of the light source unmodified by selective reflection or absorption, and also from the point of view of their relative merit when it is desired to obtain correct orthochromatic rendering of colored objects. Values relating to the photographic efficiency of these light sources are obtained by using three types of film representing the three typical classes from the standpoint of color sensitivity.

THE LATTICE ENERGIES AND PHOTOCHEMICAL DECOMPOSITION OF THE SILVER HALIDES.[3]

S. E. Sheppard and W. Vanselow.

CONSIDERATION of the energy steps in the Born cycle for calculating the lattice energies of the silver halides leads to the following conclusions: 1. The photolysis of solid silver halide might occur directly to give halogen and metallic silver. This requires only a quantum at 800 mμ or longer, equivalent to the heat of formation of the halides from the elements. 2. This photolysis might occur indirectly, by way of separation of electrons from halide ions, followed by acceptance of the electrons by silver ions. This course would require a quantum at $\lambda = 1300$ Å., or beyond, to disrupt the lattice; then a quantum at $\lambda = 3000$ Å., or beyond, to liberate electrons. In subsequent reactions of electron acceptance, *etc.*, energy might be freed, either as radiation quanta or by radiationless collisions, so that the final difference of energy equals the heat of formation. In this case primary absorption of two quanta appears necessary at thresholds much higher than those known to be operative in the photochemical decomposition of the silver halides. The contradiction might be dismissed by abandoning the view that the photolysis involves the intermediate liberation of photoelectrons from halide ions—a hypothesis suggested independently by K. Fajans and S. E. Sheppard. The phenomena of photoconductance definitely point to a relative freeing of photoelectrons. F. C. Toy has shown (*Nature*, Sept. 24,

[3] Communication No. 360 from the Kodak Research Laboratories and published in *J. Phys. Chem.* **33**: 250 (1929).

1927) that the discrepancy between the photoconductance absorption spectrum of the silver halides and the spectrum photographically active is removed by consideration of the thickness factor. The authors have recently obtained independent evidence of the liberation of photoelectrons, concomitant with the production of free halogen, in a study of the photopotentials of silver—silver halide electrodes in electrolytes. The difficulty of the energy quanta required cannot be dismissed. The experimental evidence obtained with these cells led to a modified theory of the inner photoelectric effect which is consistent with Sheppard's orientation theory of photographic sensitivity and latent image formation. By considering that both the lattice energy and electron affinity are lowered at interfaces, particularly at true interfaces with conductors, it seems possible for the inner photoelectric effect to occur at the shorter wave-lengths.

THE SUPERMOLECULAR STATE OF POLYMERIZED SUBSTANCES IN RELATION TO THIN FILMS AND INTERFACES.[4]

S. E. Sheppard, A. H. Nietz, and R. L. Keenan.

THE compounds, gelatin, rubber, and cellulose esters, may be represented by the formulas: rubber, $(C_5H_8)_n$; cellulose, $(C_6H_{10}O_5)_n$; protein, $(NHCHRCO)_n$; where n indicates both an unknown magnitude and quality or kind of aggregation.

The constitutions of such substances are well contrasted between the "association" theory of Hess and Pringsheim and the "macromolecular" theory of Staudinger.

A study of the action of these high molecular bodies in thin films and at interfaces is in agreement with Staudinger's conception of extended chains of atoms held by homopolar primary valencies. The remaining interactions, *i.e.*, orientations and condensations of these chains, are effected by development of alternating polarity along the chains and consequent mutual induction. The binding forces in this case (between the chains) may vary from Van der Waal's forces to electrostatic constraints. Under the influence of

[4] Communication No. 361 from the Kodak Research Laboratories and published in *Ind. Eng. Chem.* **21**: 126 (1928).

dissolvents, the macromolecular state may be brought to incipient dissociation—a state comparable with that postulated by the associationists.

A STILL FOR LIQUIDS OF HIGH BOILING POINT.[5]

K. C. D. Hickman.

SCARCELY volatile organic liquids which decompose on ordinary vacuum distillation can be made to volatilize and condense without harm if the apparatus is suitably shaped. The hot liquid surface is maintained without the hindrance of intervening pipes within a few centimeters of the cooling surface and vacuum is secured by a Hyvac pump. Benzyl citrate, castor oil, and fatty acid esters have been distilled.

A SUBLIMATION MERCURY STILL.[6]

K. C. D. Hickman.

DISTILLATION of mercury does not remove noble metal impurities. A three-stage distillation gives sufficient separation, however, to postpone washing the mercury residues to infrequent occasions. The still is made from three interconnected alembics supported at barometric height above four reservoirs. Mercury is transferred by sublimation through a torricellian vacuum three times in its passage from the soiled to the clean reservoir. The still is electrically heated and automatic in action.

SOME EXPERIMENTS WITH BINOCULAR AND MONOCULAR VISION.[7]

E. M. LOWRY.

AN account is given of a series of investigations, in which both binocular and monocular vision were used, of the bright-

[5] Communication No. 368 from the Kodak Research Laboratories and published in *J. Opt. Soc. Amer.* **18**: 69 (1929).

[6] Communication No. 369 from the Kodak Research Laboratories and published in *J. Opt. Soc. Amer.* **18**: 62 (1929).

[7] Communication No. 380 from the Kodak Research Laboratories and published in *J. Opt. Soc. Amer.* **18**: 29 (1929).

ness sensibility of the retina. A new type of visual sensitometer is described which enables the operator to determine the brightness functions of the eye in a variety of ways. The instrument consists of a hemispherical bowl, one meter in diameter, equipped with an illuminating system which will give brightnesses from 0.01 to 600 millilamberts. This bowl serves as the sensitizing field. A mechanism is provided by means of which the threshold for brightness may be determined at any adaptation level. Contrast sensibility may also be measured at any brightness as well as the effect on threshold and contrast sensibility of introducing various objects into the visual field. In addition, monocular and binocular acuity may be determined by using the Ives form of acuity meter. Data and curves are given to represent the results secured for several observers. Emphasis is laid on the necessity of determining the various functions of the eye binocularly if they are to be correctly interpreted in practice.

The Rational Utilization of Coal. This important industrial problem has been recently discussed at some length at a meeting of the Northeast Coast Institution of Engineers and Ship Builders, of Newcastle-on-Tyne. Papers were read by Mr. W. J. Drummond and Dr. W. T. K. Braunholtz.

A most interesting and important point is the account in Drummond's paper of the manner in which Germany has recovered its control of the fuel necessary for its industries. The peace treaty took the coal mines, assigning the Upper Silesian to Poland and the western to France. After careful investigation, the lignite deposits within German territory, which have long been considered as of little importance for power purposes, have been utilized efficiently and Germany is now using these deposits as one of the chief sources of power production. The uneconomic result of the peace treaty is shown by the fact that Poland has no large use of coal within its borders and, therefore, has to seek export methods if the mines are to be worked. As the transit to shipping ports covers about 400 miles of railway, it has been necessary to subsidize the railroad largely in order to enable the mine owners to compete with the outside sources of coal.

The papers also describe in much detail the many methods of what may be called the indirect use of coal, among which is the low-carbonization procedures, which seem to promise furnishing a smokeless fuel of high heating power. Pulverized coal is also discussed and the obtaining of fuel-oils and use of the gas produced by the low temperature methods. It is noted that soft coal is now considered as consisting of essentially four heat-producing proximate principles: fusain, durain, clarain, and vitrain. The approximate compositions and qualities of these are set forth.

H. L.

NOTES FROM THE U. S. BUREAU OF MINES.*

TWENTY YEARS OF EXPLOSIVES TESTING.

THE Pittsburgh Experiment Station of the United States Bureau of Mines recently marked the twentieth anniversary of the official opening of that station, and coincidentally of the initiation of the work of testing explosives for permissibility for use in coal mines.

As the testing of explosives for permissibility was an entirely new undertaking and one which involved passing on the merits of the products of established manufacturers, there were some apprehensions as to its outcome, states Dr. Charles E. Munroe, chief explosives engineer of the Bureau of Mines. These doubts were speedily removed through such prompt and active cooperation of the leading explosives manufacturers that, within a year from the date of the opening, it was announced, through the published lists, that 31 different explosives, made by 11 different manufacturers, had passed the tests and were approved for use in blasting in coal mines.

The interest shown by the coal mine operators was equally gratifying, for, almost immediately, the substitution of permissible explosives for those previously employed in mining coal began (cautiously at first since they must needs be used in a definitely prescribed manner to which the miner must become habituated) and they met with such acceptance that during the year 1912 the total sales of Permissible Explosives in the United States amounted to 24,630,270 pounds, of which 18,150,618 pounds were used in coal mines. From then on there has been a general annual increase in the quantities of permissibles sold (though naturally fluctuating with the fluctuations of the coal mining industry as affected by war or labor conditions) until, in 1926, the quantity of permissibles used in the United States was, during that year, 67.685,416 pounds, of which 65,143,319 pounds were used in coal mines. Owing to conditions in the coal mining industry, through which

* Published by permission of the Director, United States Bureau of Mines.

there was a stoppage of mining in many mines, the total amount of permissibles used in 1927 fell to 63,847,161 pounds and the indications now are that the figure for 1928 may not differ greatly from that last mentioned.

When the testing of explosives for permissibility was under consideration it was known that there were many factors involved about which little or no precise knowledge existed. It was true that for upwards of a quarter of a century prior to 1908 the governments of several countries, and notably those of England, France, Belgium and Germany, had been testing explosives for use in coal mines, but there was no agreement among them as to details of methods or criteria of safety, though the procedure of each tended to detect and eliminate the most unfit of the explosives which had been used or were offered for use.

Consequently in testing for permissibility it was determined to conform so far as possible, both in materials used and in methods, to the actual practices followed and conditions which then obtained in coal mining in the United States. A notable example of this was the decision to use *natural* gas rather than *manufactured* gas in all those tests in which gas was to appear as a factor, since natural gas was the particular gas encountered in American coal mines. And there were other differences from foreign testing practices which were quite as radical.

To make sure the procedure adopted was justified and to gain more precise knowledge as to the characteristics, behaviors and best use of explosives the Explosives Division of the Bureau of Mines has constantly been engaged during these twenty years in researches on these problems at the testing station, in the laboratory and in the field, the results of these researches having been embodied in many publications and, as applied, they have led to many changes in procedure in testing and use and in the compositions of permissible explosives.

Because of these, the general progress in the explosives art, and the necessity of finding substitutes for important components owing to shortages growing out of war-time and trade conditions, during these twenty years, 633 different explosives have been submitted for tests for permissibility, of

which 372 passed. For various reasons 252 of those which passed have been removed or transferred, so that on December 31, 1928, the number remaining on the Active List was 130.

The chief features that may be noted in the general development of Permissible Explosives during this period are (1) The introduction of components that depressed the freezing point, thereby producing what are styled "low-freezing," or "L.F." explosives, through which the menace from the use of frozen explosives, or in the thawing of frozen explosives, has been eliminated: (2) The greatly increased use of ammonium nitrate as the oxidizing component of explosives, this substance having been found to possess characteristics rendering it especially well suited to the production of permissibles: (3) The substitution of nitrated sugars or of nitrated glycols for nitroglycerin: (4) The reproportioning of the components so as to produce explosives giving on explosion the minimum of poisonous and inflammable gases, together with the recognition that the paper in which the explosive is wrapped must be considered as one of these components: and (5) the development of the bulky or "high-count" cartridges.

Besides modifications in the form of apparatus or method of procedure or of preparation of materials with which to conduct the tests, there have been invented and perfected the pendulum friction device with which to determine the sensitiveness of explosives to glancing blows; the sand test bomb with which to determine the relative efficiencies of detonators; and the Crashaw and Jones apparatus by which information is obtained as to the kind and quantities of the substances occurring in the products of the explosion of an explosive.

Although the investigation of explosives for use in mining coal was the primary object sought in the organization of this work, explosives for use in other mining operations, in quarrying and in engineering operations have been the subject of many suitability tests. All classes have been studied in field use, as well as in the laboratory and at the testing station, and the safest and most efficient methods of employing them pointed out. The results of some of these investigations are presented in Bureau of Mines reports entitled "The effect of stemming on the efficiency of explosives," "The effect of

cartridge diameter on the strength and sensitiveness of certain high explosives," "The selection of explosives used in engineering and mining operations" and many other publications from the Division.

In the Chemical Section new methods of analysis are the constant subject of study while many new explosive substances especially those which, as initiators, may be used in substitution for mercury fulminate have been the subject of critical investigations. Since liquid oxygen explosives have certain obvious characteristics of value, they also have been the object of extensive investigation and their limiting characteristics developed.

The Explosives Division of the Bureau of Mines has rendered expert services to various divisions of the Federal and State governments and to municipalities. From the outset it has served as the explosives inspection service of the Panama Canal Commission. Through its efforts in demonstrating the methods for their safe and efficient use in industrial operations more than 126,000,000 pounds of the military explosives left as a surplus on the conclusion of the Great War have been used in governmental engineering operations and agricultural projects. As an inspection corps the officials of this Division have investigated a large proportion of the more serious accidents and outrages involving explosives or explosions which have occurred during the past twenty years.

In 1928 a distinct novelty was introduced when the blasting device, styled "Cardox," was approved for permissibility and placed upon the Active Permissible List. This device consists of a cylindrical steel tube having an aperture in its base which may be closed by a steel disk of desired thickness and shearing strength. On the interior of the shell, but enclosed in a tube, is a quantity of a combustible material which may be ignited electrically from without. With this "heater element" inclosed and the aperture at the bottom of the shell covered by its steel disk properly secured, the shell is charged with the determined quantity of liquid carbon dioxide. Leading wires through which to transmit the electric current being attached, the closed shell is placed in the bore hole in the coal, stemmed and tamped and the current turned on, thus igniting the composition in the heating element, which pro-

duces in burning sufficient heat to raise the pressure of the carbon dioxide to such an extent that it shears a hole through the disc, and escaping through the aperture thus made pushes down the coal. For over fifty years there have been repeated attempts made to utilize the expansion of carbon dioxide in mining coal. This seems to be the first practical solution of the problem.

USE OF CAUSTIC SODA, CAUSTIC POTASH AND SODALIME IN OXYGEN BREATHING APPARATUS.

An investigation conducted by the laboratories of the Drägerwerk at Lübeck, Germany, has shown that the absorption of carbon dioxide by solid caustic soda or caustic potash is paralyzed, so to speak, when traces of certain organic vapors as gasoline, benzol, and alcohol are present in the gas mixture. This finding is of considerable importance in the use of caustic soda and potash in self-contained oxygen breathing apparatus to remove carbon dioxide from the exhaled air. If the carbon dioxide is not removed it will accumulate in what is known as the breathing bag and aside from causing undesirable physiological response as increased respiration and headaches, it will exclude the oxygen and rapidly cause collapse from oxygen deficiency—often without warning symptoms. The seriousness of such a situation may be appreciated when it is considered that the wearer of this type of apparatus is usually encountering an atmosphere that is irrespirable due to oxygen deficiency, poisonous gases, or both.

When ordinary precaution is taken the chances for contamination of the oxygen in the breathing bag or the caustic are not great when the apparatus is used in mines. Briefly, the main sources of trouble would be, contamination of the caustic during manufacture, storage and use of the material; cleaning of parts of the apparatus with paralyzing agents and the sterilizing of mouth pieces with alcohol. Also, as a small portion of alcohol taken in beverages is excreted through the lungs and more may be added to the breath by that remaining in the mouth, contamination may occur from that source. When the self-contained breathing apparatus of the type using

caustic is employed for exploring tanks, tankers, and other places where benzol, gasoline, or other similar hydro-carbons may occur, the danger from these paralyzing agents is greater. In addition to the above-mentioned sources they may gain access by permeating the rubber parts of the apparatus after long exposure, by faulty or leaky mouth pieces, by the wearer inhaling around the mouth piece or through the nose, or through carelessness in upkeep and place of storage.

Caustic soda or potash is not, however, widely used in this country for the purpose previously described. The reagent commonly used in breathing apparatus to absorb carbon dioxide is a specially prepared granular compound composed of a mixture of sodium hydroxide and calcium hydroxide. In view of the results reported by the Drägerwerk Laboratory and the chemical similarity of this compound to the ordinary caustic soda or potash, the Pittsburgh Experiment Station of the United States Bureau of Mines is making experiments in order to ascertain whether the same paralyzing phenomena is true for sodalime. Results of experiments made with the most widely used brand have so far shown that the vapors of ethyl alcohol, gasoline, or benzol have no effect on the efficacy of sodalime for removing carbon dioxide.

UTILIZATION OF WASTE NATURAL GAS.

The utilization of the enormous quantities of natural gas which are going to waste in localities where the gas cannot be disposed of as fuel, is a problem of great importance. The Pittsburgh Experiment Station of the United States Bureau of Mines has undertaken the preparation of a series of reports dealing with the economics of possible processes for converting this natural gas into valuable products which can be economically used and transported to their respective markets. Natural gas, of course, cannot be economically transported great distances except in pipe lines. However, liquid or easily liquefiable products such as formaldehyde, ammonia, methanol, and motor fuel can be made from natural gas. The proposed reports will indicate which of these products can be most profiitably prouced at the locations where natural gas is available.

DETERMINING THE CAPACITY OF NATURAL GAS WELLS.

The method in common use for determining the capacity of a natural gas well is responsible for a large waste of gas and, in many cases, serious damage to the producing capacity of the well, as it is allowed to "blow" unrestricted for fifteen minutes in order that the maximum flow of gas can be determined. It is evident that a method of determining the potential capacity of a well without allowing it to blow wide open would be of great economic value.

The Petroleum Experiment Station, maintained jointly by the United States Bureau of Mines and the State of Oklahoma at Bartlesville, Oklahoma, has therefore started an investigation of the fundamental relations involved in the flow of gas through gas sands and the relationships between gas flow and well capacity. Studies of over 400 gas wells located in different parts of the country have been made and the data correlated. The results of the investigation up to date indicate that a fairly simple and practical field method for determining the capacity of a gas well can be evolved.

An Investigation of the Absorption Spectra of Water and Ice, with Reference to the Spectra and the Major Planets. J. C. MCLENNAN, R. RUEDY and A. C. BURTON. (*Proc. Royal Soc.*, A 785.) Slipher in 1909 photographed the spectra of sunlight reflected from Jupiter, Saturn, Uranus and Neptune. All the spectra thus obtained showed absorption bands superimposed on the usual solar spectrum. Some bands may be due to atomic hydrogen in the atmospheres of the planets. The intensity and width of the absorption bands increase in general from Jupiter to Neptune. There have been three notable bands awaiting identification, (1) At 5,430 A.U., weak in the spectra of Jupiter and Saturn but very strong in Uranus and Neptune. This is in the green part of the spectrum. (2) At 6,190 A. U. Conspicuous for all four planets, broadening from a width of 50 A.U. in Jupiter to 200 A.U. in Neptune. (3) A strong double band 7,200 to 7,260 A.U., recorded only for Jupiter and Saturn. Slipher held that this last band was in the right position to be the strongest absorption band of water vapor but the other bands due to water vapor were not present on the plates.

Since a continuous spectra had never been photographed through thicknesses of water the three authors proceeded to do this, especially as there was some reason to believe that the band at 6,190 A.U. might be due to this substance. A continuous spectrum was photographed through 1, through 4 and finally 21.5 meters of water. Plates were sensitized with the same dye as that used by Slipher. While the comparison of planetary and artificial absorption spectra is far from showing identity in all details it seems probable that the second and third bands are due to water. The work of Coblentz has shown the surface temperature of the major planets to be as low as $-140°$ C. Since ice could exist at this temperature it seemed worth while to photograph the absorption spectrum of this solid. Blocks of ice about a meter long, with clear ends, were placed in a row and light was sent through as much as 14 meters of ice. The photographs of the transmitted light show marked differences from those obtained through water." There is evidently a considerable shift of the absorption bands toward the longer wave-lengths, there being in ice very little absorption where, with water, the absorption is very strong."

"It is concluded that the absorptions in the red common to the four major planets can be attributed to water, not as vapor or as ice, but in the liquid state. The absorption in the green at 5,430 Å. cannot be attributed to water and its identity is unknown. Search is being made for it in the absorption of certain liquefied gases."

G. F. S.

THE FRANKLIN INSTITUTE.

STATED MONTHLY MEETING, MARCH 20, 1929.

The regular monthly meeting was called to order by the President, Mr. Nathan Hayward. He announced that the minutes of the preceding monthly meeting had been published in full in the March number of the Journal of the Institute and that unless objection were uttered, the minutes would be approved as printed. No objection was offered, so the President ruled the minutes approved.

The Secretary reported the following changes in membership since the last regular meeting:

New members: one life member; six resident members; four non-resident members; one transfer from resident to resident life membership.

He reported, with sincere regret, the death of three members—Messrs. C. B. Cochran of West Chester, Pa.; S. P. Hutchinson of Philadelphia; and Edward Lupton of Philadelphia.

There was no further business except the reading of the paper of the evening, so the President introduced Dr. S. L. Hoyt of the General Electric Company, who spoke on "The New Alloy, Carboloy; Its Development and Uses." The lecturer first of all showed a moving picture film which displayed the use of the new alloy 'carboloy' in the industries. He then outlined the development of this alloy and dwelt particularly upon the upbuilding of the new art involved in its production. Dr. Hoyt's interesting and important paper was received with much enthusiasm and was followed by an animated discussion. The meeting adjourned at ten-eight p.m., with a rising vote of hearty thanks to the lecturer.

HOWARD MCCLENAHAN,
Secretary

COMMITTEE ON SCIENCE AND THE ARTS.

(Abstract of Proceedings of Stated Meeting held Wednesday, March 6, 1929.)

HALL OF THE INSTITUTE,
PHILADELPHIA, March 6, 1929.

DOCTOR THOMAS D. COPE *in the Chair.*

A second reading of the report of the Special Committee on the Revision of the Regulations of the Committee on Science and the Arts was presented by Mr. Franklin, Chairman, and after discussion was adopted.

The following reports were presented for final action:

No. 2896: The Work of Mr. Elmer A. Sperry.

The Sub-Committee of Investigation decided to limit its field to those inventions that are based as was Mr. Sperry's former invention of the Gyro Compass on the gyroscope. These include a Gyro-Pilot which automatically steers a vessel upon a prescribed course; a Gyro-Stabilizer which largely eliminates the roll of a vessel; a Gyro Roll and Pitch Recorder which permits a comparison to be made

between the motion of a vessel with and without the stabilizer; and a Gyro Track Recorder which enables the irregularities in a railroad track to be located and measured.

All of these inventions have been put into extensive practical use and have proved most helpful in their respective fields.

The award of the Elliott Cresson Medal was recommended to Mr. Sperry "In consideration of his application of the principle of the gyroscope to the development of means whereby vessels may be navigated with greater safety, comfort, and economy, both in time and fuel, and whereby the irregularity of movement of vessels and trains may be accurately recorded."

No. 2901: Self-Aligning Coupling.

The Fast Flexible or Self-Aligning Coupling is a device for coupling together the ends of a driving and driven shaft in such a way as to permit the transmission of power with a minimum of loss.

The device consists of a coupling that does not make use of any flexible material but fastens to the end of each shaft a gear, the teeth of which mesh into the internal teeth of a pair of collars that are bolted together over the ends of the coupled shafts.

The form of the teeth is such as to permit a considerable misalignment of the shafts. The casing containing the collars retains a quantity of oil that is distributed to the gear teeth by the centrifugal force of rotation.

The award of the John Price Wetherill Medal to Mr. Gustave Fast, of Annapolis, Maryland, was recommended, "In consideration of the novelty of design of this coupling, the stability of its construction and of its successful practical application."

The following reports were presented for first reading:

No. 2856: Vapor Phase Process.

No. 2906: Literature.

GEO. A. HOADLEY,
Secretary to Committee.

MEMBERSHIP NOTES.

ELECTIONS TO MEMBERSHIP.

(*Stated Meeting, Board of Managers, March 13, 1929.*)

LIFE.

MR. RANDAL MORGAN, III, Student, St. Paul's School, Concord, N. H.

RESIDENT.

MR. ALFRED THOMAS BRATTON, Patent Attorney, Fraley and Paul, 1815 Land Title Building, Philadelphia, Pa. For mailing: 1008B West Erie Avenue, Philadelphia, Pa.

MR. JOHN PIM CARTER, Research Chemist in Manufacturing, 3144 Passayunk Avenue, Philadelphia, Pa. For mailing: 314 Maple Avenue, Drexel Hill, Pa.

DR. EDWARD HILL COX, Professor of Chemistry, Swarthmore College. For mailing: 8 Whittier Place, Swarthmore, Pa.

DR. DUNCAN GRAHAM FOSTER, Teacher, Department of Chemistry, Swarthmore College. For mailing: 504 Walnut Lane, Swarthmore, Pa.

MR. FRANK C. ROBERTS, JR., Taylor-Wharton Iron and Steel Company, Widener Building, Philadelphia, Pa.

MR. ANDREW SIMPSON, Engineer, Swarthmore College, Swarthmore, Pa.

NON-RESIDENT.

MR. DEWITT D. BARLOW, Civil Engineer, President, Atlantic Gulf and Pacific Company, 15 Park Row, New York City.

MR. JOHN J. CATON, JR., Textile Engineer, Wallace Silk Company, Phillipsburg, N. J. For mailing: 306 Mercer Street, Phillipsburg, N. J.

MR. ISAAC HARTER, Engineer and Manufacturer, Vice-President, Babcock and Wilcox Company, 85 Liberty Street, New York City.

MR. JAMES P. WALLACE, President, Wallace Silk Company, Phillipsburg, N. J. For mailing: 306 Mercer Street, Phillipsburg, N. J.

TRANSFERS.

FROM RESIDENT TO RESIDENT LIFE.

MR. MORRIS CLOTHIER, Merchant, Philadelphia.

CHANGES OF ADDRESS.

MR. WILLIAM DEKRAFFT, Fidelity-Philadelphia Trust Building, Broad and Walnut Streets, Philadelphia, Pa.

MR. CHARLES F. FORSTALL, Reading Company, Spring Garden Street Station, Philadelphia, Pa.

MR. OTTO T. MALLERY, 225 South Fifteenth Street, Philadelphia, Pa.

MR. EDWARD W. PARKER, Room 607, Morris Building, 1421 Chestnut Street, Philadelphia, Pa.

MR. G. R. REBMANN, 502 North Nineteenth Street, Philadelphia, Pa.

MR. A. T. SMITH, 565 Park Avenue, New York City.

MAJOR JOHN A. VOGELSON, 2031 Locust Street, Philadelphia, Pa.

MR. W. CHATTIN WETHERILL, Chestnut Hill Post Office, Chestnut Hill, Philadelphia, Pa.

NECROLOGY.

J. Snowden Bell

J. Snowden Bell was born in Philadelphia on July 11, 1843, and died in Brooklyn, N. Y., on November 27, 1928. He was educated in the public schools of Philadelphia, the Polytechnic College, and the University of Pennsylvania, receiving from the latter institution the degree of Bachelor of Laws in 1879.

In 1862 he entered the service of the Baltimore and Ohio Railroad as draftsman but soon after received the appointment of assistant engineer on the U. S. Frigate Minnesota, serving in this capacity, with distinction, to the end of the Civil War.

In 1865 he returned to the Baltimore and Ohio shops as chief draftsman where he supervised the designing of several of the passenger and freight engines. For a brief period he was engineer and superintendent of the Pittsburgh Locomotive and Car Works and about 1870 opened an office in Philadelphia as consulting mechanical engineer, and solicitor of patents.

In 1879 Mr. Bell was first admitted to the bar—being admitted to practice before the Philadelphia Bar on June 14, 1879; in all he was admitted to practice before some 23 courts, including the United States Supreme Court.

From 1883 to 1909 his office for the practice of patent law was in Pittsburgh, Pa. In the latter year he removed to New York where he continued his patent practice. Mr. Bell was an authority on locomotive boilers, front ends, superheaters, feed-water heaters, etc., frequently contributing papers on these subjects to the several associations with which he was connected. He also wrote numerous papers on the history and development of locomotives and is also noted for his work on patent law known as "Bell on Expert Testimony." He was an active member of the Master Mechanics Association, the mechanical section of the American Railway Association, Engineers' Club of New York, and numerous other engineering organizations.

He became a member of The Franklin Institute on March 10, 1875.

Mr. C. B. Cochran, West Chester, Pa.
Mr. S. Pemberton Hutchinson, Philadelphia, Pa.
Mr. Edward Lupton, Philadelphia, Pa.

LIBRARY NOTES.

RECENT ADDITIONS

Archivio di Storia della Scienza. Volumes 1–8. 1919–1927.

BAKER, CLYDE. Modern Gunsmithing. 1928.

BRADY, GEORGE S. Materials Handbook. First edition. 1929.

CLOUD, JOHN WILLS. Castles in the Ether. Revised edition. 1928.

DAHL, O. G. C. Electric Circuits: Theory and Applications. Volume one: Short-Circuit Calculations and Steady-State Theory. First edition. 1928.

DUNCAN, RICHARD. Air Navigation and Meteorology. Second edition. 1928.

ELLIS, CARLETON, AND JOSEPH V. MEIGS. Gasoline and Other Motor Fuels. 1921.

FAIRBANKS, ERNEST E., editor. The Laboratory Investigation of Ores: a Symposium. First edition. 1928.

FLETCHER, HARVEY. Speech and Hearing. 1929.

FLETTNER, ANTON. The Story of the Rotor. 1926.

FREEMAN, JOHN R., Editor. Hydraulic Laboratory Practice. 1929.

HOLLAND, MAURICE, AND HENRY F. PRINGLE. Industrial Explorers. 1929.

HUMPHREYS, W. J. Physics of the Air. Second edition, revised and enlarged. 1929.

International Critical Tables of Numerical Data, Physics, Chemistry and Technology. Volume 5. 1929.

Jahrbuch für Photographie und Reproduktionsverfahren für die Jahre 1921–1927, Herausgegeben von J. M. Eder. Dreizigster Band, 1 und 2 Teil. 1928.

Klimschs Jahrbuch. Band XXII. 1929.
KUNS, RAY F. Automotive Essentials. 1928.
LEE, WILLIS T. Stories in Stone. 1926.
MACKENZIE, CATHERINE. Alexander Graham Bell: the Man Who Contracted Space. 1928.
MARTIN, GEOFFREY. A Treatise on Chemical Engineering Applied to the Flow of Industrial Gases, Steam, Water, and Liquid Chemicals. 1928.
MELLOR, J. W. A Comprehensive Treatise on Inorganic and Theoretical Chemistry. Volume IX. 1929.
OLSEN, J. K. Production Design. 1928.
ORNSTEIN, MARTHA. The Rôle of Scientific Societies in the Seventeenth Century. 1928.
Photograms of the Year 1928. Thirty-fourth year.
REINTHALER, FRANZ. Artificial Silk. Enlarged and revised edition, translated from the German by F. M. Rowe. 1928.
SEVERNS, WILLIAM H., AND HOWARD E. DEGLER. Steam, Air, and Gas Power. 1929.
STOCK, ALFRED. The Structure of Atoms. Translated from the second German edition by S. Sugden. No date.
STOTT, VERNEY. Volumetric Glassware. 1928.
TINKLER, C. K., AND HELEN MASTERS. Applied Chemistry. Volume I. 1929.
TOMMASINA, THOMAS. La Physique de la Gravitation et la Dynamique del' Univers. 1928.
TRAVERS, MORRIS W. The Discovery of the Rare Gases. 1928.
VALIER, MAX. Raketenfahrt. 1928.
WIEN, W. Das Wiensche Verschiebungsgesetz. Ostwald's Klassiker der exakten Wissenschaften. Nr. 228. 1929.
World Almanac and Book of Facts for 1929.

BOOK REVIEWS.

APPLIED CHEMISTRY. By C. Kenneth Tinkler, D.Sc., F.I.C., and Helen Masters. B.Sc. Volume I, second edition revised. xii–296 pages, illustrations, 12mo, London, Crosby Lockwood & Son, 1929. Price, 15 shillings.

Applied chemistry has a wide significance. It may comprehend the vast field of modern industry, or the manifold phases of medicine. The book in hand is a text-book for students in Household Science and Public Health. It contains many tests and quantitative processes, and is, of course, a compilation from standard literature with occasional additions from the personal experiences of the authors. It is a second edition and the first volume of a series. It relates to: Water, Detergents, Textiles, and Fuels with some minor topics. Volume II is announced as covering Foods, being written by the same authors.

The information offered is of standard character, and does not call for detailed criticism. Emanating from British sources, one notes not unexpectedly, for instance, that in the procedure for the determination of nitrates in water, the extensive and important work done some years ago in the United States concerning the phenol-sulphonic method is entirely ignored. Similarly, the

delicate method for nitrites with alpha-aminonaphthalene is unnoticed. The routine given for water analysis is, however, full enough to secure judgment of its potability and industrial applications according to generally accepted standards.

In the section on textiles, the familiar photomicrographs of the more important raw fibres are given, but no indication of the degree of magnification. Photographs are also given to show the effects of treating mixed textiles for determination of the several fibres. They are of no use. The only proper method for instruction in such work is to let the student carry out the processes and make the examination under the microscope. Textiles of known composition should be submitted and the student obligated to make the analysis.

A reference to "rayon" gives the information that efforts to have the British call this by some name other than "artificial silk" have so far failed. Within a comparatively small space, making a handy volume, a large amount of information is given, and the careful use of the book will serve to introduce the student to much material of practical application. It is not stated how many volumes will constitute the series, but it is evident that a very comprehensive treatment of some phases of applied chemistry is in view.

HENRY LEFFMANN.

LA GOMME DE BALATA. By A. D. Luttringer. Pamphlet, 50 pages, 12mo. Paris, A. D. Cillard, 1929.

Balata is a material, closely resembling gutta-percha, derived from the latex of a tree *Mimusops globosa* Gaert. of the Sapotaceae, the family to which our common persimmon belongs. The tree grows widely in northern South America especially in the Guianas. The pamphlet gives a full account of the methods of extraction of the material, its purification, composition and adaptability as a substitute for ordinary gutta-percha for which it seems to be highly suitable.

H. L.

PHYSICS OF THE AIR. By W. J. Humphreys, C.E., Ph.D. Second edition, revised and enlarged. 654 pages, illustrations, 8vo. New York, McGraw-Hill Book Company, Inc., 1929. Price, $6.

Doctor Humphreys' comprehensive volume covered at its first appearance a wide and interesting field. Its success has led to a second edition, in the preparation of which the author has made extensive revisions and considerable additions, among the latter especially noted a chapter on meteorological acoustics, grouping many of the sounds due to the relation of the air in motion to obstructing conditions, as observed for instance in the soft and soothing murmuring of the forest or the terrifying noise of the tempest. Much information is also given as to the laws of transmission of sound through air in different conditions. A very recent development that might be classified under the above title is the relation of weather conditions to radio transmission. The weather bureau has begun to advise the "fans" concerning probabilities in this matter.

The book is liberally illustrated with maps and graphs, also apparatus. It would have been worth while to insert one of the several available excellent pictures of the funnel-shaped cloud constituting the tornado, and, perhaps a couple of simple maps showing the approximate course and width of recent tornadoes, of which, indeed the middle west of this country has had sad experi-

ence. Pictures of the destruction wrought by such storms are of no particular interest.

Doctor Humphreys has considerable of the saving grace of humor and writes in a more or less colloquial style. His large experience and intimate acquaintance with subjects of the work have been fully utilized in its text. It is to be hoped that someone equally qualified will give us a book on the *chemistry* of the air. Many years ago a most interesting and valuable series of lectures appeared in the Chemical News on "The Supposed Nature of Air, Prior to the Discovery of Oxygen."

The reviewer feels that the canons of criticism would not be fully observed, if the general character of the text was passed over without some animadversion. There is certainly a good deal of carelessness in the literary form. Many sentences would be likely to secure a flogging for Macauley's learned school-boy of the fifth form. There is a good deal of indifference to proper phraseology now noticed in American books. It is possibly another expression of determination to have "personal freedom." We consider Lindley Murray as much a trespasser on our privileges as Volstead.

It is gratifying to see a work of such scientific character find a hearty response from those to which it is addressed. In a future edition some specific notice might be given of the peculiar "ricochet" effect observed in the operation of sirens and fog-horns along the coast. HENRY LEFFMANN.

LA PHYSIQUE DE LA GRAVITATION ET LA DYNAMIQUE DE L'UNIVERS, BASÉES SUR LA DÉCOUVERTE DU MÉCANISME RÉEL DES RADIATIONS DÉMONTRANT LEUR FONCTION GRAVITATIONNELLE RÉCIPROQUE. Par Thomas Tommasina, Docteur ès Sciences. vi–302 pages, 8vo, paper. Paris, Gauthier-Villars et Cie, 1928. Price, 50 francs.

Most books that analyze the fundamental concepts of physical science employ every generally known mathematical device (and often some not generally known) in their demonstrations, the argument being interwoven with analytical deduction that imposes upon the reader a concentration upon mathematical transformations more or less at the expense of a desirable reflective attitude in viewing the subject. Doctor Tommasina employs the exceptional course reminiscent of Cornot's classic, Reflections on the Motive Power of Heat; it is wholly philosophical and reflective. The aim of the book is to pass in critical review the concepts of modern physics and to furnish a guide, sure, precise and understandable of the purely mechanical and material nature of all the phenomena in the field of astronomy and physics.

In an introduction, the author makes a summary review of the evolution and present state of modern physics. He then proceeds with a lengthy consideration of the fundamental notions upon which scientific deduction may be based. An equally lengthy account upon when and how the physics of universal gravitation was initiated and developed includes quotations of the works of many of the most celebrated physicists of modern times which are in agreement with his own views. Next to pass in review is the work of that famous leader in modern mathematical physics, H. Poincare and of a celebrity of a century earlier J. H. Lambert as forerunners in the theory of relativity. A discussion of the work of Einstein under the title "A Einstein, his two relativist theories and his erroneous

interpretation of physical phenomena" occupies a chapter of satisfying length, and as the title indicates, of a distinctly critical character. A note on the recent work of Jean Chazy on relativity, also critical, and a list of the author's 119 scientific memoirs from 1899 to 1914 conclude the work.

The work is a timely contribution toward bringing modern views of the fundamental laws of physical science into concordance with observed phenomena which should prove of inestimable value as a reference for investigators in the fundamental principles of physics.

L. E. P.

RAKETEN-FAHRT (ROCKET-TRAVEL). By Max Valier, fifth edition. 252 pages, illustrations, 8vo. Munich and Berlin, R. Oldenbourg, 1928. Price, in paper 8.50 marks, bound 9.50 marks.

"Glücklicher Säugling! dir ist ein unendlicher Raum noch die Wiege; Werde Mann und dir wird eng die unendliche Welt."

Schiller's epigram might well be the motto of this book, for it expresses the modern Alexandrine desire to pass from the conquests that have been made in this world to the worlds beyond. Wide notice has already been given of Valier's exploits with the rocket-driven auto, and a short time ago, newspapers presented discussions of Goddard's suggestion for a rocket shot at the moon. That such proposals are met with doubts and even with open and severe condemnation need not astonish us. Such distinguished scientists as Lord Kelvin and Simon Newcomb deprecated Langley's efforts and expressed doubts as to whether a machine heavier than air could be raised by its own power.

Valier was the first and fearless institutor of a peculiar method of automobile propulsion, and is enthusiastic as to its wide application, considering it not only available for surface travel, but entirely feasible for aviation and even for sailing through the interplanetary space. He would probably modify our well-known English phrase and say "No pent-up Utica contracts our powers, but the whole boundless universe is ours."

The book is a comprehensive and extended discussion of the problem, throughout most optimistic. Theoretical as well as practical questions are to be solved. The preface opens with a romantic account of a telescope study of Mars in 1924 when its approach to the earth was unusually close. The preliminary discussion on the relations of gravity and interplanetary conditions must be left to Einstein and his disciples. The reviewer is quite out of his depth.

Much information is given concerning the make-up of rockets for the several purposes for which they are used. A historical note exhibits rather over-trustful acceptance of early allusions, among other statements, the suggestions that Moses and his contemporaries were familiar with high explosives and incendiary methods of warfare. It is, of course, well known that a work purporting to be written by "Marcus, the Greek," entitled "The Book of Fires to burn Enemies" dates from at least as early as the Eighth century and probably derived from much earlier oriental sources. Many instances of fiery projectiles employed in war are given, the definite invention of Congreve in 1799 being discussed.

The possibility of travelling through interplanetary space is discussed more seriously than would be expected, but the author is naturally deeply interested in the question. About sixteen pages are given to the "Theory of Travelling in

the Cosmos" and considerably more space to the difficulties and interferences connected with these ventures. A doubter might be inclined to ask what would the traveller to the moon do when he got there? Surely there is not to be expected any possibility of "re-fueling." In our common experience a rocket goes up brilliantly and rapidly in defiance of gravitation, but it comes down as a stick under the ordinary law of acceleration, except as influenced by the resistance of the air.

Undoubtedly this book will awaken much thought on the part of those who are striving after new and startling performances. There is no limit apparently to human ambition. The book is liberally illustrated and excellently printed. The literary style is typically German, many of the sentences being long and involved, but owing to the rule as to the locating of adjective phrases, German lends itself to such construction more than any of the familiar languages. The fact that in four years five editions have been published shows the wide-spread interest in the subject. HENRY LEFFMANN.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

Report No. 296. Pressure Distribution Tests on PW-9 Wing Models From − 18° Through 90° Angle of Attack, by Oscar E. Loeser, Jr. 21 pages, illustrations, quarto. Washington, Government Printing Office, 1929, price, fifteen cents.

At the request of the Army Air Corps, an investigation of the pressure distribution over PW-9 wing models was conducted in the atmospheric wind tunnel of the Committee. The primary purpose of these tests was to obtain wind-tunnel data on the load distribution on this cellule to be correlated with similar information obtained in flight tests, both to be used for design purposes. Because of the importance of the conditions beyond the stall as affecting control and stability, this investigation was extended through 90° angle of attack. The results for the range of normal flight have been given in N. A. C. A. Technical Report No. 271. The present paper presents the same results in a different form and includes, in addition, those over the greater range of angle of attack, − 18° through 90°.

The results show that

At angles of attack above maximum lift, the biplane upper wing pressures are decreased by the shielding action of the lower wing.

The burble of the biplane lower wing, with respect to the angle of attack, is delayed, due to the influence of the upper wing.

The center of pressure of the biplane upper wing (semispan) is, in general, displaced forward and outward with reference to that of the wing as a monoplane, while for the lower wing there is but slight difference for both conditions.

The overhanging portion of the upper wing is little affected by the presence of the lower wing.

Report No. 297. The Reduction of Observed Airplane Performance to Standard Conditions, by Walter S. Diehl. 27 pages, illustrations, quarto. Washington, Government Printing Office, 1928, price, fifteen cents.

This report shows how the actual performance of an airplane varies with air temperature when the pressure is held constant. This leads to comparatively simple methods of reducing observed data to standard conditions. The new

methods which may be considered exact for all practical purposes, have been used by the Navy Department for about a year, with very satisfactory results.

The report also contains a brief historical review of the important papers which have been published on the subject of performance reduction, and traces the development of the standard atmosphere.

Report No. 314. Drag and Cooling with Various Forms of Cowling for a "Whirlwind" Radial Air-Cooled Engine—II, by Fred E. Weick. 22 pages, illustrations, quarto. Washington, Government Printing Office, 1929, price, twenty cents.

This report gives the results of the second portion of an investigation in the Twenty Foot Propeller Research Tunnel of the Committee, on the cowling and cooling of a "Whirlwind" J-5 radial air-cooled engine. The first portion, which is reported in N. A. C. A. Technical Report No. 313, pertains to tests with a cabin fuselage. This report covers tests with several forms of cowling, including conventional types, individual fairings behind the cylinders, individual hoods over the cylinders, and the new N. A. C. A. complete cowling, all on an open cockpit fuselage. Drag tests were also made with a conventional engine nacelle, and with a nacelle having the new complete cowling.

In the second part of the investigation the results found in the first part were substantiated. It was also found that the reduction in drag with the complete cowling over that with conventional cowling is greater with the smaller bodies than with the cabin fuselage; in fact, the gain in the case of the completely cowled nacelle is over twice that with the cabin fuselage. The individual fairings and hoods did not prove effective in reducing the drag The results of flight tests on an AT-5A airplane (reported in the Appendix to N. A. C. A. Technical Report No. 313) have been analyzed and found to agree very well with the results of the wind tunnel tests.

R.

BIBLIOGRAPHY OF AERONAUTICS, 1926. Published by the National Advisory Committee for Aeronautics. 154 pages, 8vo. Washington, Government Printing Office, 1928, price, twenty cents.

BIBLIOGRAPHY OF AERONAUTICS, 1927. Published by the National Advisory Committee for Aeronautics. 184 pages, 8vo. Washington, Government Printing Office, 1928, price, thirty-five cents.

These volumes cover the aeronautical literature published during 1926 and 1927. The first Bibliography of Aeronautics was published by the Smithsonian Institution as Volume 55 of the Smithsonian Miscellaneous Collections and covered the material published prior to June 30, 1909. Supplementary volumes of the Bibliography of Aeronautics for the subsequent years have been published by the National Advisory Committee for Aeronautics. They are the Bibliography of Aeronautics for the years 1909–1916, 1917–1919, 1920–1921, 1922, 1923, 1924, 1925.

As in previous volumes, citations of the publications of all nations are included in the languages in which these publications originally appeared. The arrangement is in dictionary form with author and subject entry, and one alphabetical arrangement. Detail in the matter of subject reference has been omitted on account of the cost of presentation but an attempt has been made to give sufficient cross reference for research in special lines. R.

YEAR BOOK OF THE AMERICAN PHARMACEUTICAL ASSOCIATION, 1926, volume 15, containing the Sixty-Ninth Annual Report on the progress of pharmacy, and the Constitution, By-laws and Roll of Members. lxxi–613 pages, illustrations, 8vo. Baltimore, American Pharmaceutical Association, 1928.

The series to which this volume belongs is well-known in Pharmacy, and is framed upon the principles now general in such reports. The value of a publication of this type will depend upon the extent to which the literature has been examined and the judgment by which the abstracts have been selected. Collaborators will always differ as to what is suitable and what not, but taken all in all the collection here presented is a broad and comprehensive summary of the progress of pharmacy during the year 1926. It seems regrettable that the volume is so late, for more than two years have elapsed since the literature abstracted was closed.

A detailed examination is impossible, not only for lack of time and space (an Einsteinian would probably say "space-time") but comparison of the abstracts with the originals would be necessary. The reviewer turns naturally to a topic with which he has some familiarity practically as well as from the literature. The detection of methanol in the presence of ethanol has become a very important analytic problem in the United States for obvious reasons. The abstracts here given on this subject have not been collated. American chemists and pharmacists have been for several years urging that the term "methyl alcohol" should be abandoned to avoid any popular notion that the substance has use as a substitute for common alcohol. It is the duty of abstractors to support this effort or if they fail the editor should take care of the matter. Yet "methyl alcohol" is prominently indexed with two references to page 196 and "methanol" with reference to page 346. There is some confusion in the statements in the several paragraphs, and a lack of definiteness in some parts. Thus, referring to Kolthoff's method of detecting methanol in presence of ethanol the phrase is "(oxidation of the methyl alcohol by phosphoric acid solution and detection of the resulting formaldehyde by means of Schiff's reagent) is more sensitive than the method of Denigès (oxidation by means of acid permanganate)". Phosphoric acid is not an oxidizing agent and the standard procedure of the U. S. P. X. is by permanganate in solution made acid by phosphoric acid. Later in the text the fact that the phosphoric and sulphuric acid methods are mere variants is indicated, but the group of three abstracts on the detection of this important adulterant of common alcohol leaves the matter in some confusion. No mention is made of the fact ascertained several years ago by LaWall that glycerol will simulate methanol in the test and the liquid to be tested should therefore always be distilled.

HENRY LEFFMANN.

STEAM, AIR, AND GAS POWER. By William H. Severns, M.S., Member A.S.M.E., Associate Professor of Mechanical Engineering, University of Illinois, and Howard E. Degler, M.E., M.S., Member A.S.M.E., Assistant Professor of Mechanical Engineering, University of Illinois. vii–425 pages, illustrations, 8vo, cloth. New York, John Wiley & Sons, Inc., 1929. Price $4.

College courses which deal with apparatus for the production of power from fuel cover an exposition of the principles and a description of the construction and operation of an innumerable variety of appliances. A descriptive treatise in

which representative types of such apparatus are discussed with an account of the principles of their operation is necessarily an essential feature of such a course of study. It is not so long ago that texts on that subject, were not altogether satisfactory in representing current practice. Besides progress goes on apace and when new appliances and methods come into use text-books must follow the procession. In that connection, it may be noted that 150 lbs. was a high boiler pressure in 1883 while 1800 lbs. is the present day record, and in large units. Since the early years of the century books of that sort have been getting better and better, and this one in point of scope, arrangement and quality of execution is well within the "limits of error of observation" in that progressive improvement. The field covered is extensive, and to successfully condense into one volume of moderate dimensions the "necessary and sufficient" proclaims rare discrimination on the part of the authors.

The work covers a progressive study of the power-plant, the boiler and auxiliaries and the engine. Water-tube boilers predominate as is consistent in this era of high pressures. The auxiliaries include economizers, feed-water heaters and purifiers, water softening, air preheaters; it is all very complete and in excellent form. Among the engines, we have the high-speed, corliss, poppet-valve unaflow and a separate chapter on the principles and structure of the steam turbine. Pumps, compressed air and compressors, and internal-combustion engines each have an analytical and descriptive chapter. The theory of each type of apparatus is fully considered with ample descriptions and their 262 superior cuts and diagrams. Besides, there are chapters in the beginning of the book on thermodynamic principles, steam calorimetry, fuels and combustion; the first named containing a surprising amount of applicable theoretical information in a comparatively small space.

The work will doubtless satisfy the requirements of many engineering courses as well as meet the needs of numerous other readers who seek accurate information on the apparatus employed in power generation and their principles of operation.

LUCIEN E. PICOLET.

INFRA-RED ANALYSIS OF MOLECULAR STRUCTURE. By F. I. G. Rawlins, M.Sc., Trinity College, and A. M. Taylor, M.A., late scholar of Trinity College, Ramsay Memorial Research Fellow. 176 pages, illustrations, 8vo. New York, The Macmillan Company, 1929.

This is the first volume of a new series of monographs on Physical Chemistry to be prepared under the auspices of the Cambridge Press. This volume is devoted to data obtained by means of infra-red spectroscopy. This part of the spectrum seems to have attracted, so far, little detailed attention compared to that which has been given to the ultra-violet, the main reason being probably that the latter rays are more brilliant in their visible optical effects, more widely applicable in some departments of research and more useful in medical science. The existence of rays beyond the red of the visible spectrum was discovered by Herschel, but for many years not much attention was given to them. Wood made some interesting observations, especially the fact that ordinary leaves throw off considerable infra-red, as is shown in his well-known picture of a tree in full leaf, appearing as if covered with snow. The discovery made a few years ago that some flowers reflect ultra-violet light introduces us to a remarkable color contrast which

we cannot formulate, as both rays are invisible to us. If they are visible to any of the insect tribe, great aid to distinction of leaf and flower will be given.

The general scientist, that is, one who has not specialized in this field turns with some curiosity to the methods of producing infra-red rays. It is generally known that they are at the "heat-end" of the spectrum, and we find that heat-producing apparatus such as a Nernst glower or a silica pencil is used. It is stated that in the early work the Welsbach mantle was used, but this is discarded. Any one who has worked with a Nernst lamp will be familiar with the heat it emits, and it appears that much of this is of the type useful in these researches. The special topic of the book is presenting in much detail, the application of the rays to study of molecular structure, presumably somewhat in the manner in which the ultra-violet is used, especially that group called conventionally X-rays. Much space is given to construction and operation of the apparatus for producing the dispersion of the rays. A remarkable effect on an ordinary photograph plate is described by which a plate slightly inpressed by visible rays is exposed to the infra-red spectrum when reversal of image occurs on development, positive images, that is, bright lines in the spectrum, are obtained. Details are given as to the method to be used to obtain this result. It has also been noted that some phosphorescent paints, especially the calcium sulphide preparations, will be affected by infra-red, in such a manner that the phosphorescence will be materially diminished if subjected to such rays and even by visible red rays.

The data presented in the book are of a highly specialized character and correspondingly recondite, but a very large amount of information is presented. It is the first book in English devoted to the special subject and will be a welcome addition to the literature of chemical physics.

HENRY LEFFMANN.

ENCYCLOPÉDIE LEAUTÉ. LES COMBUSTIBLES LIQUIDES ARTIFICIELS. By A. Mailhe, Professeur a la Faculté des Sciences de Paris. 280 pages, illustrations, 8vo. Paris, Gauthier-Villars & Cie, Masson & Cie., 1929. Price 30 francs.

This is a summary of chemical and physical data concerning the many forms of combustible liquids available for heat and power, the main motive in the making of the volume being the wide-spread interest in obtaining substitutes for natural petroleum. The Paris Faculty of Sciences has established recently a chair for the study of combustibles and the author of the book, who has devoted himself to extensive researches in this field, has been appointed to the position. The subject-matter covers a wide field, including, of course, both aliphatic and cyclic compounds. The methods and effects of "cracking" are discussed at length. The English word has been admitted without change of spelling into the French language. It would be interesting to know how it is pronounced by our Gallic friends. However, this is merely a little persiflage. Apart from the combustible materials themselves, accessory compounds are often described. The utilization of lignite, oil-bearing shales, and products of redistillation of tar receive considerable notice.

We, therefore, have in this work, an excellent summary of the many sources from which the world may recruit its supply of heating and power fuel if the natural sources are exhausted, a condition that is beginning to be feared by some authorities.

H. L.

STEAM TURBINES, A PRACTICAL AND THEORETICAL TREATISE FOR ENGINEERS AND STUDENTS, INCLUDING A DISCUSSION OF MERCURY AND GAS TURBINES. By James Ambrose Moyer, S.B., A.M. Sixth edition, revised and enlarged. ix–557 pages, illustrations, 8vo, cloth, with an entropy chart folder. New York, John Wiley & Sons, Inc., 1929. Price $4.50.

"Light for a million homes"—all at one time, is the striking illustrated announcement in the pictorial section of a recent issue of a New York daily by the Westinghouse interests of the capacity of one of their steam turbines installed at the Hell Gate electric power station in New York City. That a generator of such dimensions is possible is eloquent testimony of the progress which has been made in large-scale power production. A considerable number of years have passed since the supremacy of the turbine over the piston engine in economy, capacity and cost became established for large outputs. During that period the engineering profession has been as generous in supplying constructive literature to that development as it was in prior years in dealing with piston engines. Professor Moyer has an honored place among these contributors as engineer, teacher and author. His "Steam Turbines" first appeared in 1908, and its present issue in a sixth edition is ample evidence of its value in effecting its intended purpose.

The work, in general, covers principles, design and descriptive matter of the types of turbines which have survived the test of performance. The discussion of the subject is preceded by a short historical survey, the elementary theory of heat, and a detailed explanation of the application of the temperature-entropy diagram. The present edition has been amplified to keep pace with progressive refinements in design and construction. Particular reference may be made to new material on "bleeder" turbines, variation of steam consumption with age, shaft design and casings for high pressures. Mechanical difficulties with the early designs of reduction gears brought about the Melville-Macalpine floating-frame reduction gear. This appears to be given more exclusive prominence than is justified by its omission in recent practice, careful workmanship having apparently overcome the need of that refinement of design. With revisions and additions in keeping with recent progress, the book will doubtless continue its successful career.

L. E. P.

PUBLICATIONS RECEIVED.

Max Valier. *Raketenfahrt.* 5 Auflage von Vorstoss in den Weltenraum. Eine technische Mĕglichkeit. 252 pages, illustrations, 8vo. München und Berlin, R. Oldenbourg, 1928, price, in paper, 8.50 marks.

Physics of the Air, by W. J. Humphreys, C.E., Ph.D. Second edition, revised and enlarged. 654 pages, illustrations, folded plate, 8vo. New York, McGraw-Hill Book Company, Inc., 1929, price $6.00.

The Elements of Practical Mechanics, by Charles Ranald MacInnes. 131 pages, illustrations, 8vo. New York, D. Van Nostrand Company, 1929, price $2.25.

Applied Chemistry. A practical handbook for students of household science and public health by C. Kenneth Tinkler, D.Sc., and Helen Masters, B.Sc.

Volume I—Water, detergents, textiles, fuels, etc. Second edition, revised. 296 pages, illustrations, 8vo. London, Crosby, Lockwood and Company, 1929, price 15 shillings.

Encyclopédie du Caoutchouc et des Matières Plastiques. La gomme de Balata par A. D. Luttringer. 50 pages, 12mo. Paris, A. D. Cillard, 1929.

Statistical Mechanics. The theory of the properties of matter in equilibrium. Based on an essay awarded the Adams prize in the University of Cambridge, 1923–24, by R. H. Fowler, M.A. 570 pages, illustrations, 8vo. Cambridge, University Press, New York, The Macmillan Company, 1929, price 35 shillings, net.

Year Book of the American Pharmaceutical Association, 1926, vol. 15, containing the sixty-ninth annual report on the progress of pharmacy and the constitution, by-laws and roll of members. 613 pages, illustrations, portrait, 8vo. Baltimore, The Association, 1928.

Bibliography of Aeronautics, 1926. National Advisory Committee for Aeronautics. 154 pages, 8vo. Washington, Government Printing Office, 1928, price twenty cents.

Bibliography of Aeronautics, 1927. National Advisory Committee for Aeronautics. 184 pages, 8vo. Washington, Government Printing Office, 1928, price thirty-five cents.

CURRENT TOPICS.

The Terminal Velocity of Drops. W. D. FLOWER. (*Proc. Physical Soc.*, London, Vol. 40, Pt. 4.) Liquid drops falling through quiet air after they have descended some distance get a constant velocity. This paper has to do with the measurement of the distance fallen and of the constant terminal velocity. Incredible as it may seem the author was unable to discover any previous determinations of the distance fallen before this constant velocity is established. A drop was liberated and fell upon the pan of a ballistic torsion balance turning about a horizontal axis. The distance from the point of release to the pan was progressively increased until no greater deflection was obtained by further increase. The drop was caught upon a horizontal sheet of filter paper. A fresh sheet was used for each drop, care being taken that all sheets occupied exactly the same position. The drops fell from a burette. When the drops form slowly and break away under the force of gravity they vary but little in volume. The range of volume for drops of water was from .2 to .7 c.c. As one would expect the larger drops fell a greater distance before they attained their constant velocity. When this state has been reached the acceleration due to gravity is just equal to the retardation caused by the air. For drops having volumes of .0194, .0415 and .0688 c.c. respectively the corresponding distances fallen were 225,316 and 340 cm. Drops of methyl salicylate also were tried. This liquid has 19 per cent. greater density than water, about half its surface tension and three times its viscosity. Drops of this liquid and of water having volumes of about .04 c.c. need to fall the same distance to attain constant velocity, but for larger sizes the distance for water is greater and for smaller volumes it is greater for the other.

To determine the constant terminal velocity a photographic method was used. For drops of water having volumes of .006, .01, .0414, and .083 c.c. the velocities attained were respectively 556.7, 611.5, 774.9 and 810.7 cm./sec. All the results obtained for the salicylate are larger than those for water drops of comparable size. The author sets up formulæ involving the coefficient of resistance and the radius of the drop and from these deduces that water drops with a radius of .25 cm. and salicylate drops with a radius of .21 cm. will fall faster than those of any other dimensions. Unfortunately in the case of both liquids the radius for maximum final speed lies

near the end of the series of radii investigated and hence it is impossible to see whether there actually is a maximum speed or not. Lenard of Heidelberg has reached the same conclusion for water and his explanation "is that the friction of the air causes deformation of the drops, so that instead of retaining the shape of spheres they become flattened out, thus presenting an increased surface to the air through which they are falling. This deformation is slight for the smaller drops, but increases rapidly in the case of the larger drops. After a certain size of drop has been attained, any further increase in volume produces a greater flattening, and instead of the velocity being increased it is slightly decreased." G. F. S.

The Influence of Electric and Magnetic Fields on the Crystallization of Supercooled Liquids. W. KONDOGURI. (*Z. f. Physik*, Vol. 47, page 589.) About twenty years ago it was discovered that the crystallization of molten sulphur was hastened by exposing it to the rays coming from radium. In following a line of experiments suggested by this result Kondoguri found that the electric field acts as radium does but has a stronger effect. In the present paper he goes on with his investigations using the magnetic as well as the electric field. Two liquids, piperin and salol were examined. Piperin melts at 128° and salol at 42.5°. Both of them are quite permanent in the liquid state when cooled below the melting temperature. Films of the liquids, .05 or .06 mm. in thickness, contained between coverglasses were put under the microscope so that the centers of crystallization could be seen and counted. The direction of the electric field was parallel to the surface of the film, while that of the magnetic field was perpendicular to it. The highest voltage applied was about 420 and the magnetic field lay between 9,000 and 10,000 gauss. An electric battery supplied the difference of potential and a Dubois electromagnet the magnetic field. Salol is much more productive of results than piperin. With no external influence in operation it is true that piperin develops centers of crystallization more rapidly than salol under comparable conditions relative numbers being 15 : 2, 33 : 5, 55 : 7, 7 : 1. When, however, an electric or magnetic field is applied, piperin manifests a much smaller change than salol, the ratio being roughly 1 : 10. Thus because salol is more sensitive and in addition more stable in the supercooled state it was chiefly used.

Counts were made at intervals extending over as much as sixteen days of the numbers of crystallization centers seen through the microscope in definite portions of the supercooled liquid, *a*. exposed to radium rays, *b*. acted on by the electric field and, *c*. left to itself without external influence. The numbers of centers for these three conditions were as follows:

After interval of	2	4	6	9	11	16 days
a. Radium	3	17	51	76	79	80
b. Electric Field	23	67	118	160	176	200
c. Control Liquid	1	2	6	10	14	19

Under the action of radium or of the electric field the number of centers reaches a limit after the lapse of sufficient time. After this has been attained the application of a higher voltage causes a further increase which continues until a new limit has been reached. The number of centers is certainly greater for strong electric fields than for weak ones. The author claims proportionality between field strength and the rate of initial increase of centers.

For the study of the influence of a magnetic field piperin was used because the heat from the electromagnet melted the centers that formed in salol. Only a part of the liquid could be used for counting the centers since in other parts they formed compact groups. However it was clear that with the passage of time the number of centers increased. In the case of piperin in the electric field a variation of the normal procedure was noticed. After several days, the number of centers having almost reached the limit, there was a sudden drop. Thereupon the number rose to its former value or even exceeded it. Once there was no drop after the limit had been almost attained instead a sudden rise toward a new limit took place. G. F. S.

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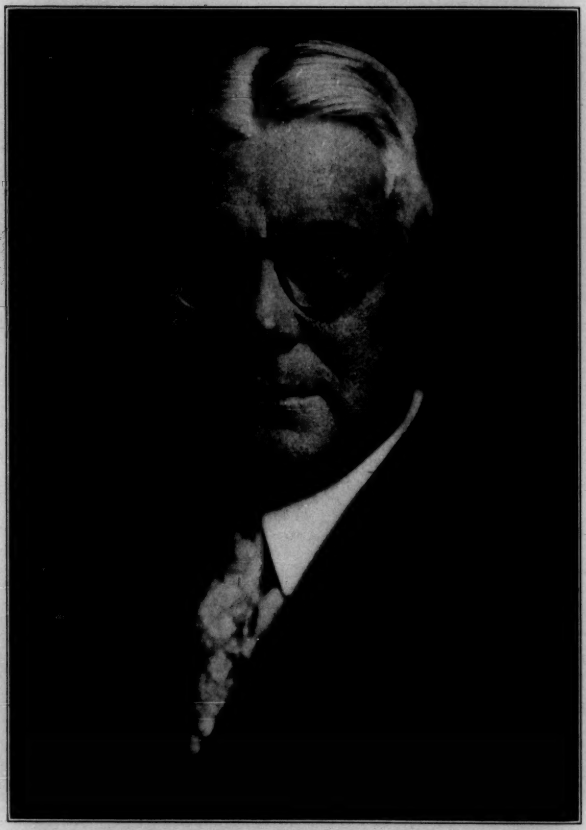

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